# Training Neural Networks with Gradient Descent

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https://dannagurari.colorado.edu/course/neural-networks-and-deep-learning-spring-2025/

# Review

- Last lecture:
  - Motivation for neural networks: need non-linear models
  - Neural networks’ basic ingredients: hidden layers and activation units
  - Neural networks’ support for diverse problems: output units
  - Objective function: what a model should learn

- Assignments (Canvas):
  - Problem set 1 due Tuesday

- Questions?

# Today’s Topics

- Gradient descent: how neural networks learn
- Mathematical foundation of gradient descent: derivatives
- Applying gradient descent to train neural networks
- Training example

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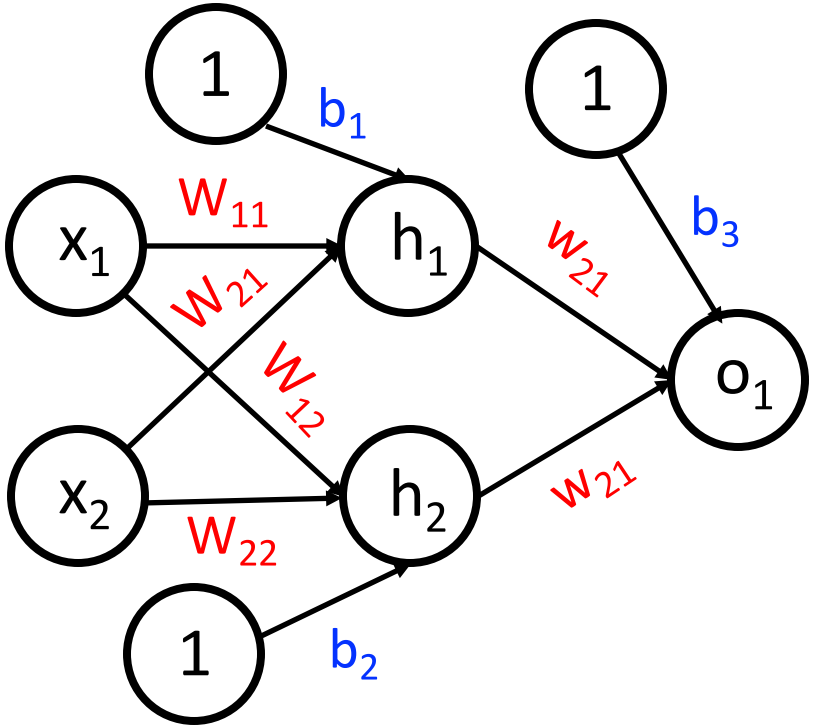
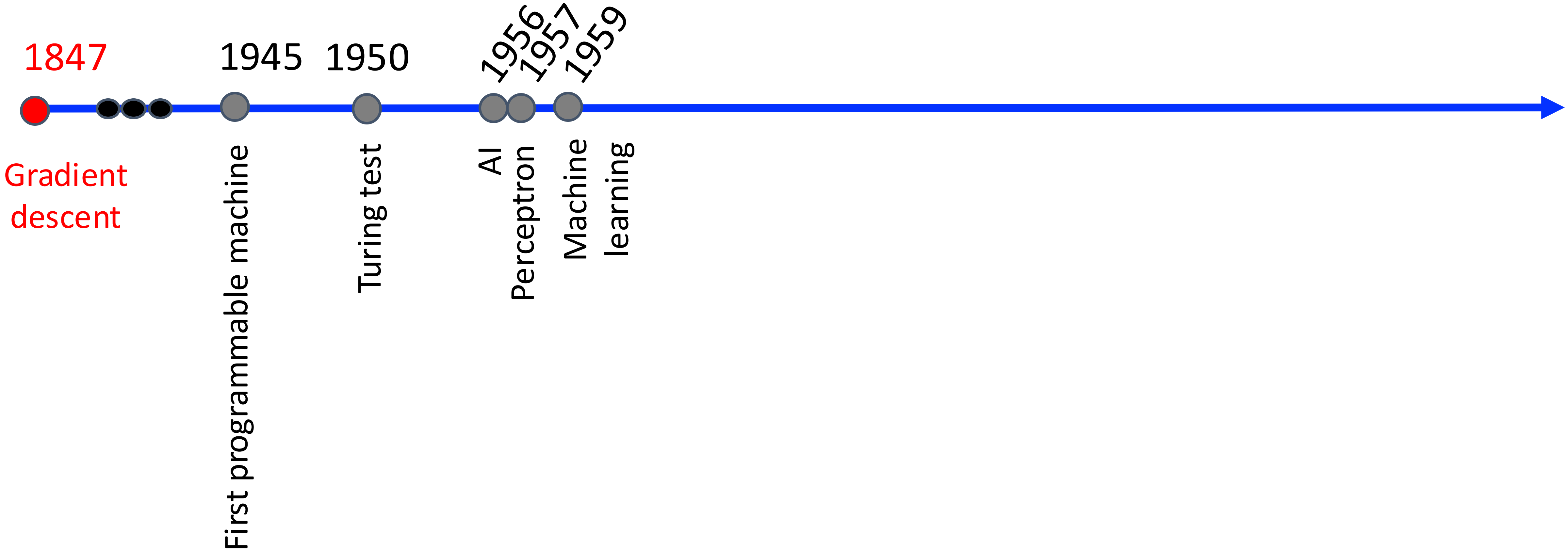
# Recall Goal: Train Neural Network to Minimize an Objective Function

- Learn model parameters that minimize an objective/loss function using gradient descent; e.g., (weights, biases)



# Approach: Gradient Descent




Louis Augustin Cauchy: Compte Rendu `a l'Acad´emie des Sciences of October 18, 1847

# Gradient Descent: Intuition

- Iteratively searching for better model parameters (e.g., weights and biases) that lead to smaller “losses” in the objective function
- Analogy: hike from mountains to Boulder blind or blindfolded!

# Gradient Descent: Intuition

- Repeat:
  1. Guess
  2. Calculate error
- e.g., learn linear model for converting kilometers to miles when only observing the input “miles” and output “kilometers”


Miles → Kilometers = miles x constant → Kilometers

# Gradient Descent: Intuition

- Repeat:
  1. $\boxed{\text{Guess}}$
  2. Calculate error
- e.g., learn constant multiplier to convert US dollars to Israeli shekels

$10 → Shekels = dollars x constant

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$10 → Shekels = dollars x constant → Error = Guess - Correct

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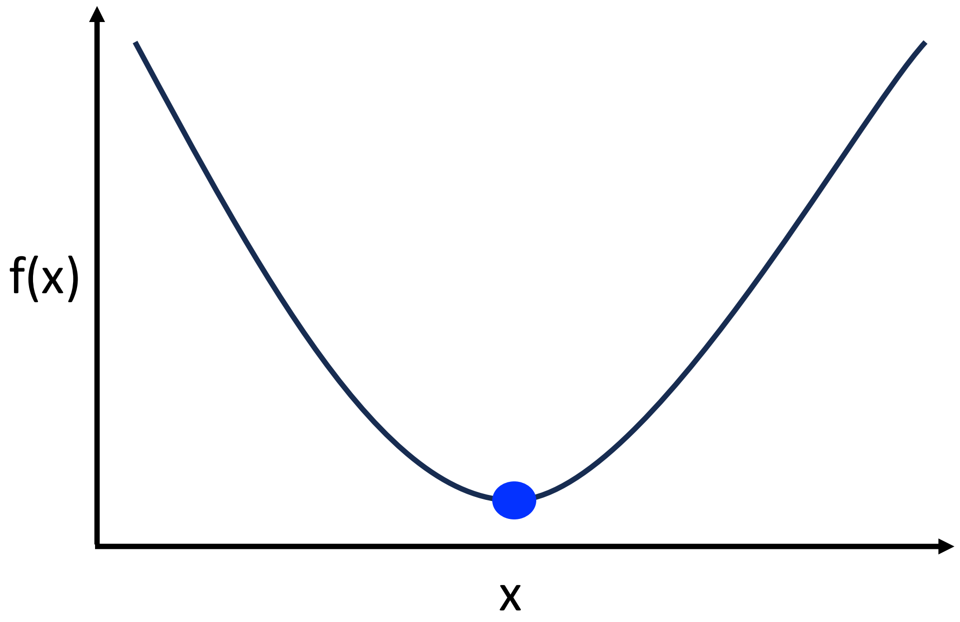
- Idea: iteratively adjust constant (i.e., model parameter) to reduce error

# Gradient Descent: Possible Scenarios

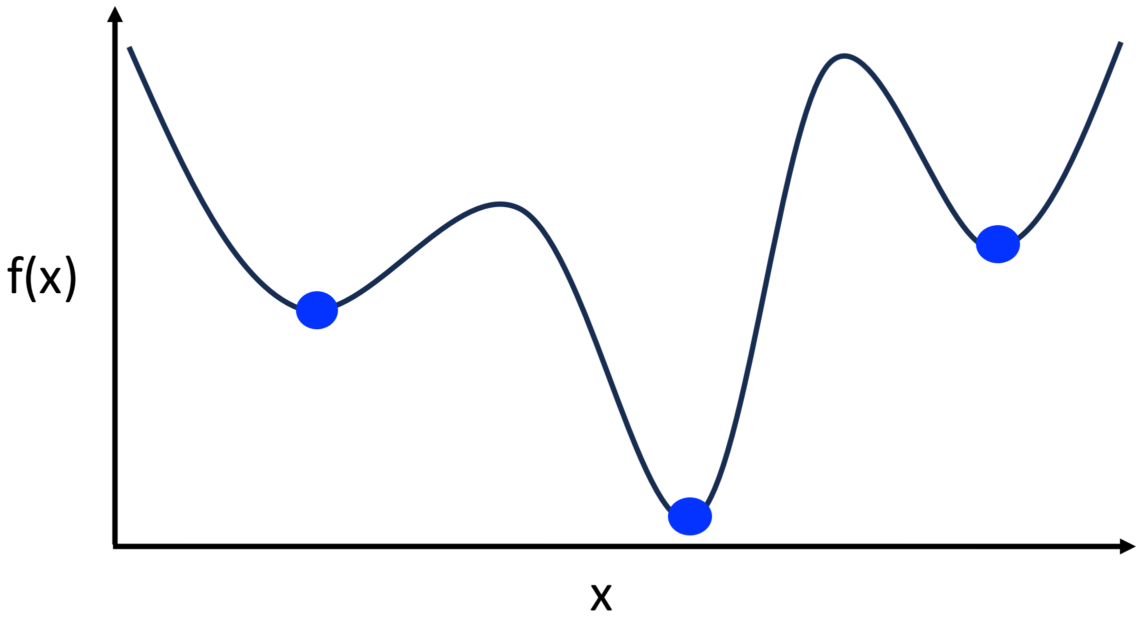
(simple 1-dimensional plots)

Convex Functions (one minimum)




Currency conversion example

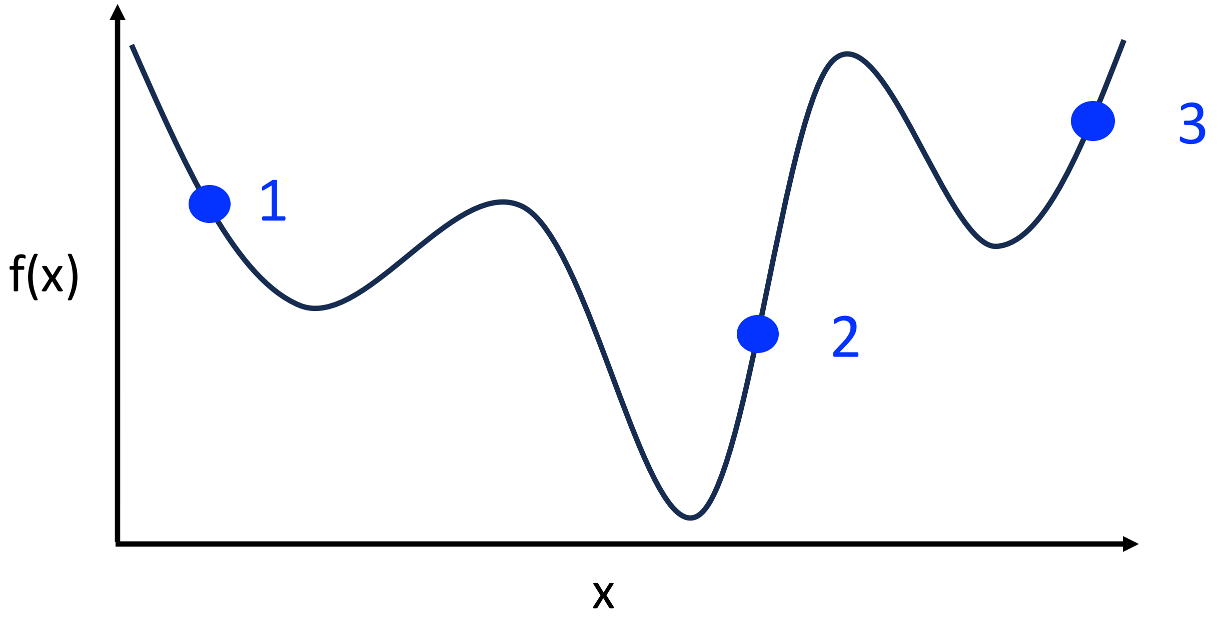
Non-Convex Functions (multiple minima)




Our focus: deep learning

# Gradient Descent: Definitions

- **Gradient:** a vector indicating how a slight change to each function variable in ***x*** increases the output ***f(x)*** (i.e., partial derivatives for multiple variables)
- Recall, a derivative indicates the slope (rise/run) of the function at any point
- **Gradient descent**: to ***minimize*** the function, iteratively step in opposite direction of gradient (i.e., descent rather than ascent)




- What does the derivative look like at position 1? (Class volunteer)
- Is position 1’s slope positive or negative?
- From position 1, should follow opposite the gradient direction to minimize the function

- What does the derivative look like at position 2? (Class volunteer)
- Is position 2’s slope positive or negative?
- From position 2, should follow opposite the gradient direction to minimize the function

# Gradient Descent: Definitions

- **Gradient:** a vector indicating how a slight change to each function variable in ***x*** increases the output ***f(x)*** (i.e., partial derivatives for multiple variables)
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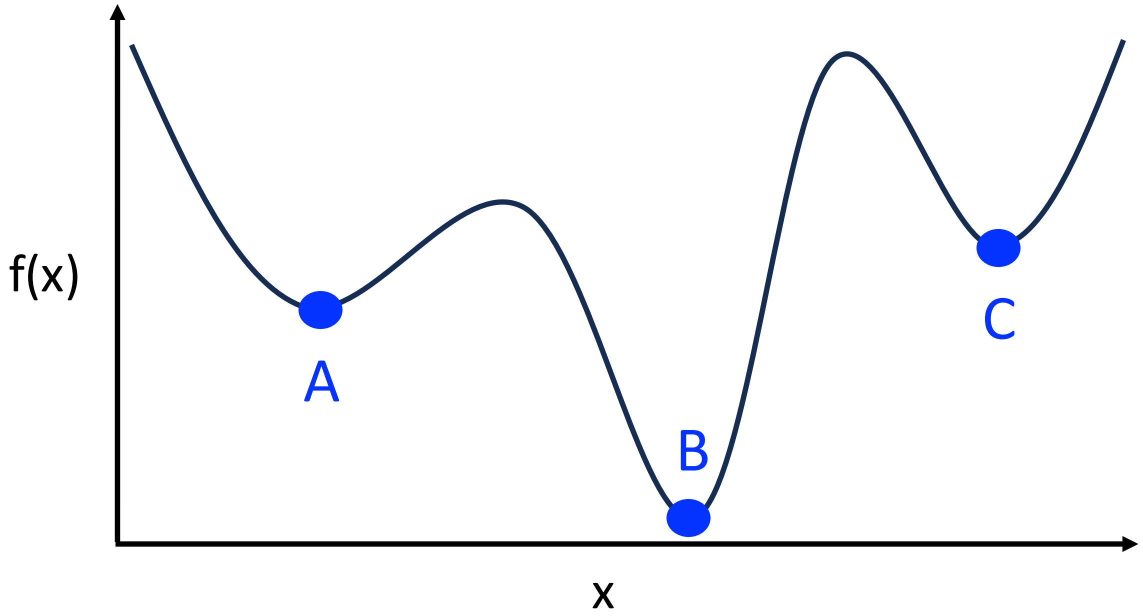
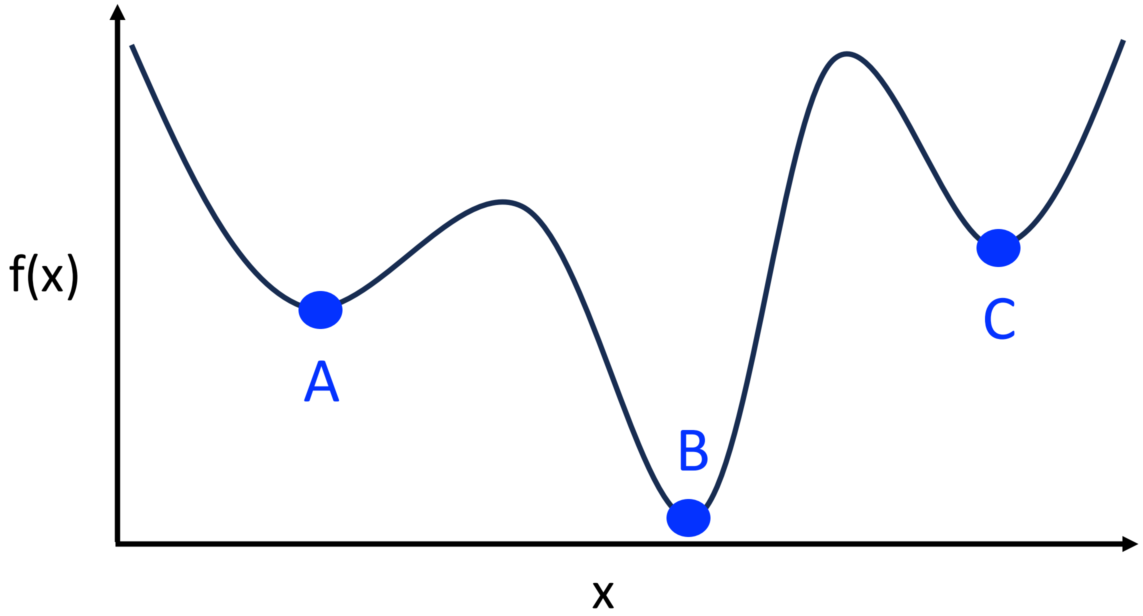



Which letter(s) show global minima?

Which letter(s) show local minima?

# Gradient descent can arrive at a local minimum rather than global minimum, a possibility that initially deterred its adoption for neural networks




Global minima: B

Local minima: A, B, C

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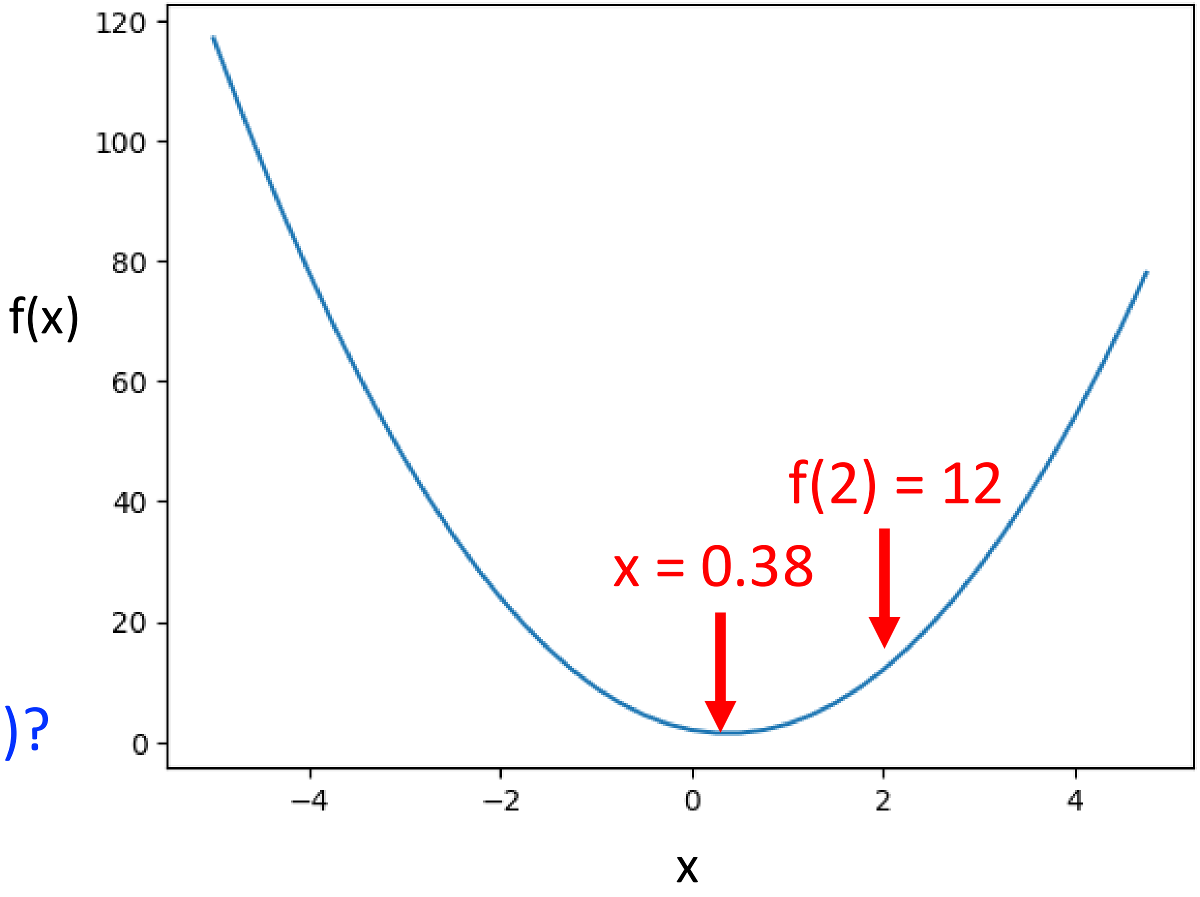
# Reviewing Fundamentals: Toy Example 1

$f(x) = 2 - 3x + 4x^2$




Is this function convex or non-convex?

What is the value of the function at x =2?

What is the value of x at the minimum of f(x)?

# Reviewing Fundamentals: Toy Example 1

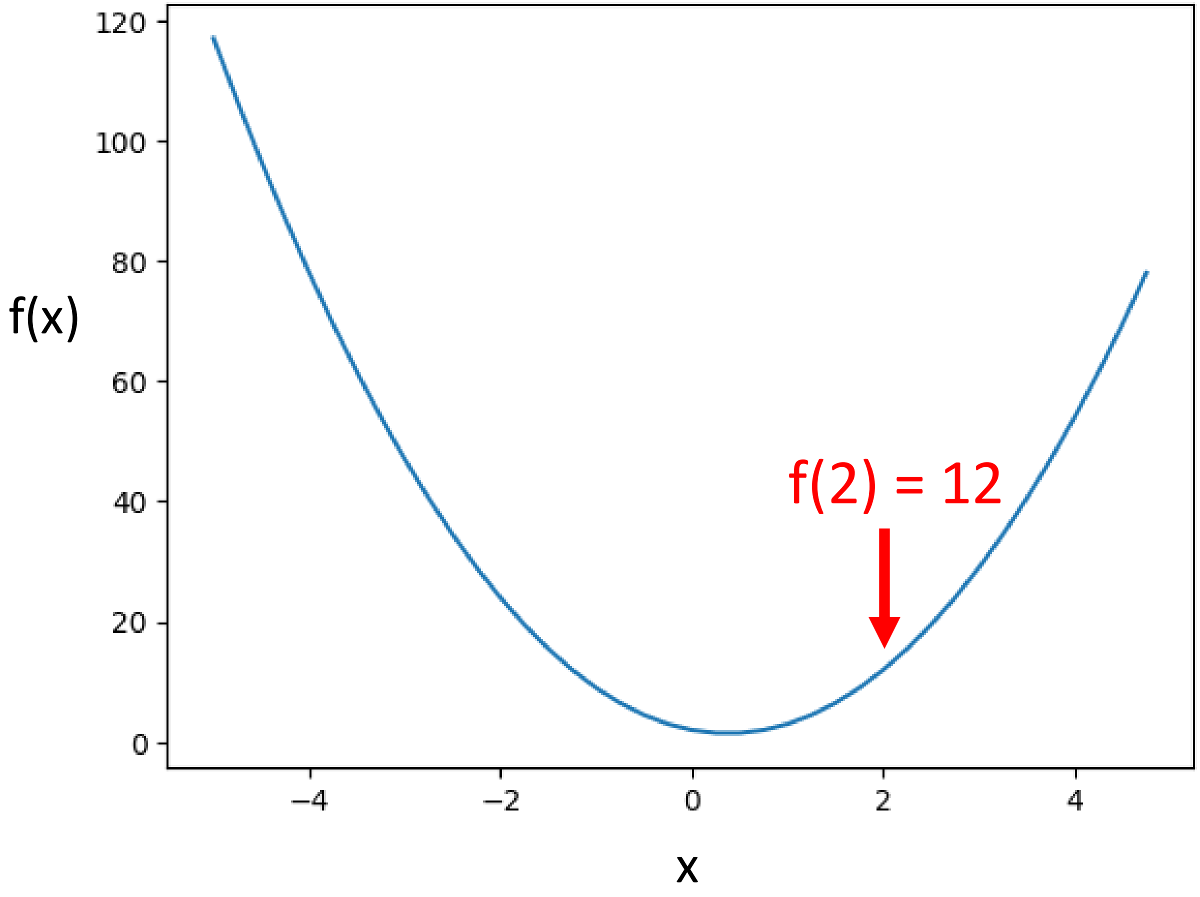
$f(x) = 2 - 3x + 4x^2$

Derivative indicates how a slight change (i.e., h) to every x will change the output:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}$$

If *h* is slightly positive, at x=2, will *f(x+h)* be greater than or less than 12?

By how much?



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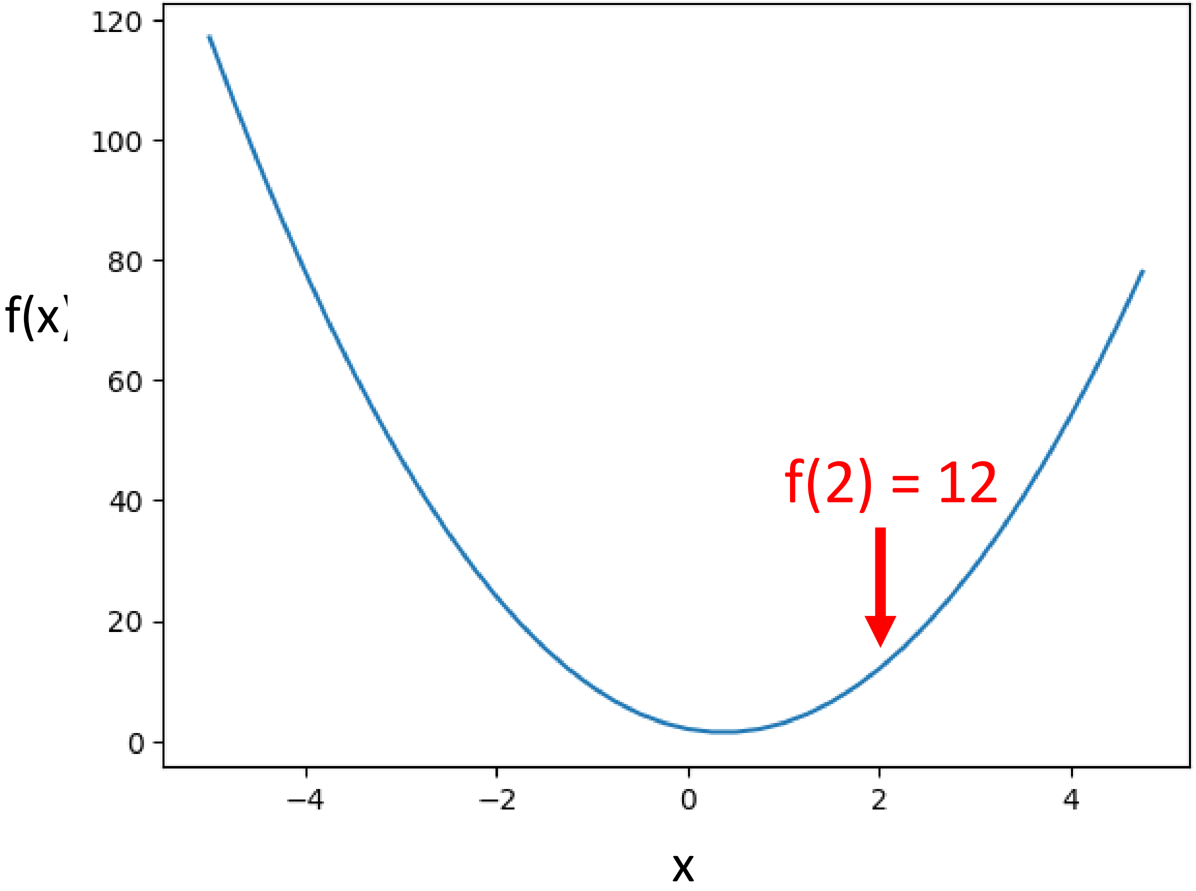
$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}$$

e.g., at $x = 2$

- for $h = 0.01$: $f(x+h) = 12.1304$, slope $= 13.04$

Increasing x by h=0.01 results in an increase of ~13.04 times that amount (13.04 * .01)



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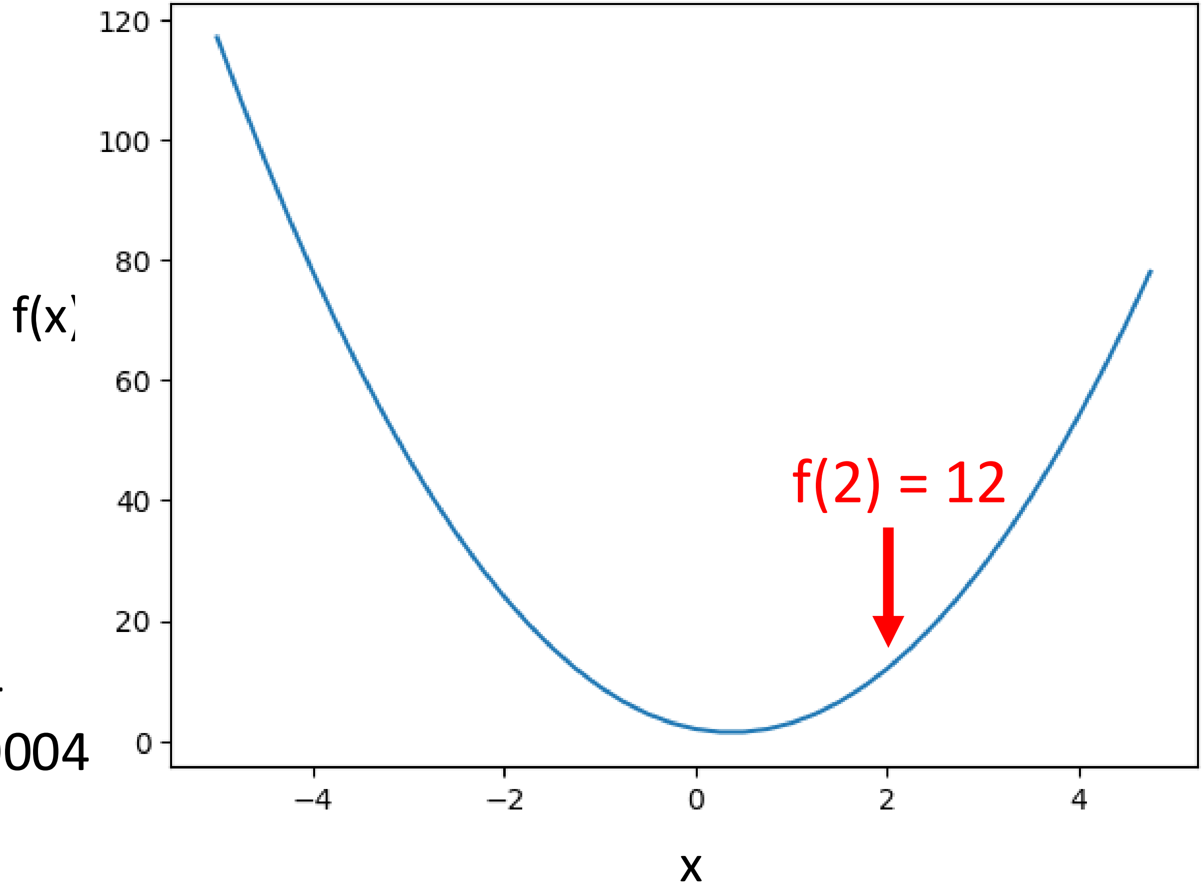



e.g., at x = 2

- for h = 0.01: $f(x+h) = 12.1304$, slope = 13.04
- for h = 0.001: $f(x+h) = 12.013004$ & slope = 13.004
- for h = 0.0001: $f(x+h) = 12.0013004$ & slope = 13.0004

As h gets closer to 0, slope approaches what?

Mathematically, $f'(x) = -3 + 8x$; $f'(2) = -3 + 16 = 13$

# Reviewing Fundamentals: Toy Example 1

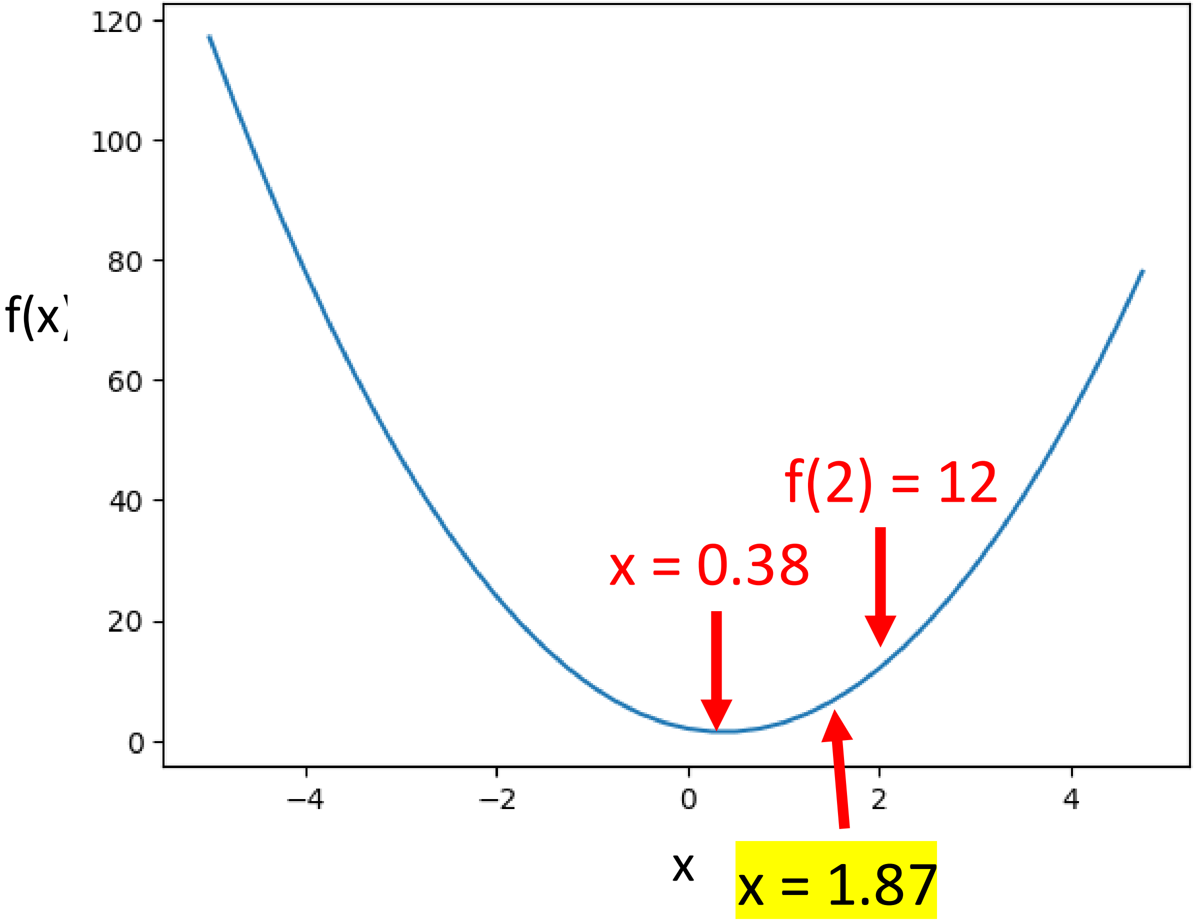
$f(x) = 2 - 3x + 4x^2$

Derivative indicates how a slight change (i.e., h) to every x will change the output:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}$$

e.g., at x = 2 and df/dx = 13
Approach minimum point by stepping in opposite direction of derivative (e.g., learning rate of 0.01); what should be new value for x?

$2 - (0.01 * 13) = 1.87$



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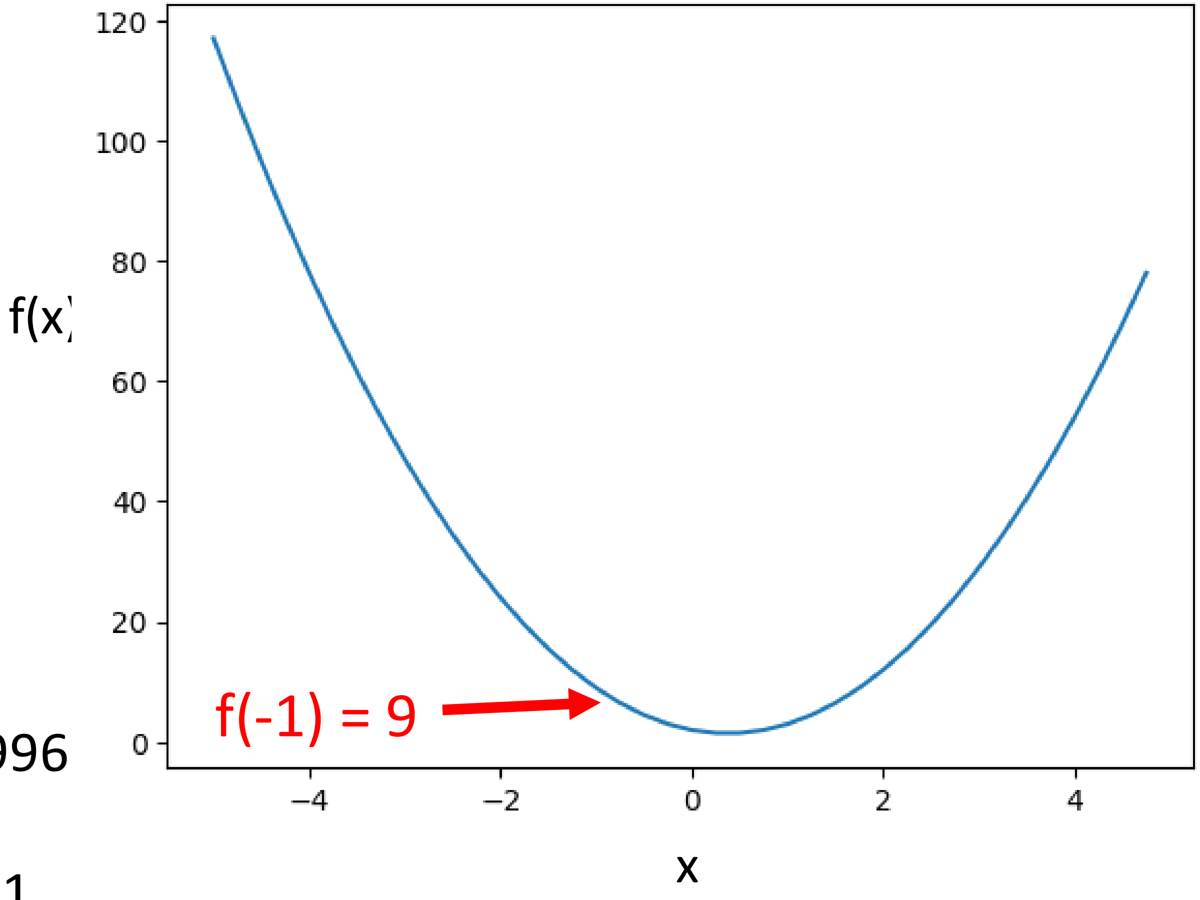



e.g., at x = -1
- for h = 0.01: f(x+h) = 8.8904, slope = -10.96
- for h = 0.001: f(x+h) = 8.989004, slope = -10.996
- for h = 0.0001: f(x+h) = 8.99890004, slope = -10.9996

As h gets closer to 0, slope approaches what?

Mathematically, $f'(x) = -3 + 8x$; $f'(-1) = -3 + 8*-1 = -11$

# Reviewing Fundamentals: Toy Example 1

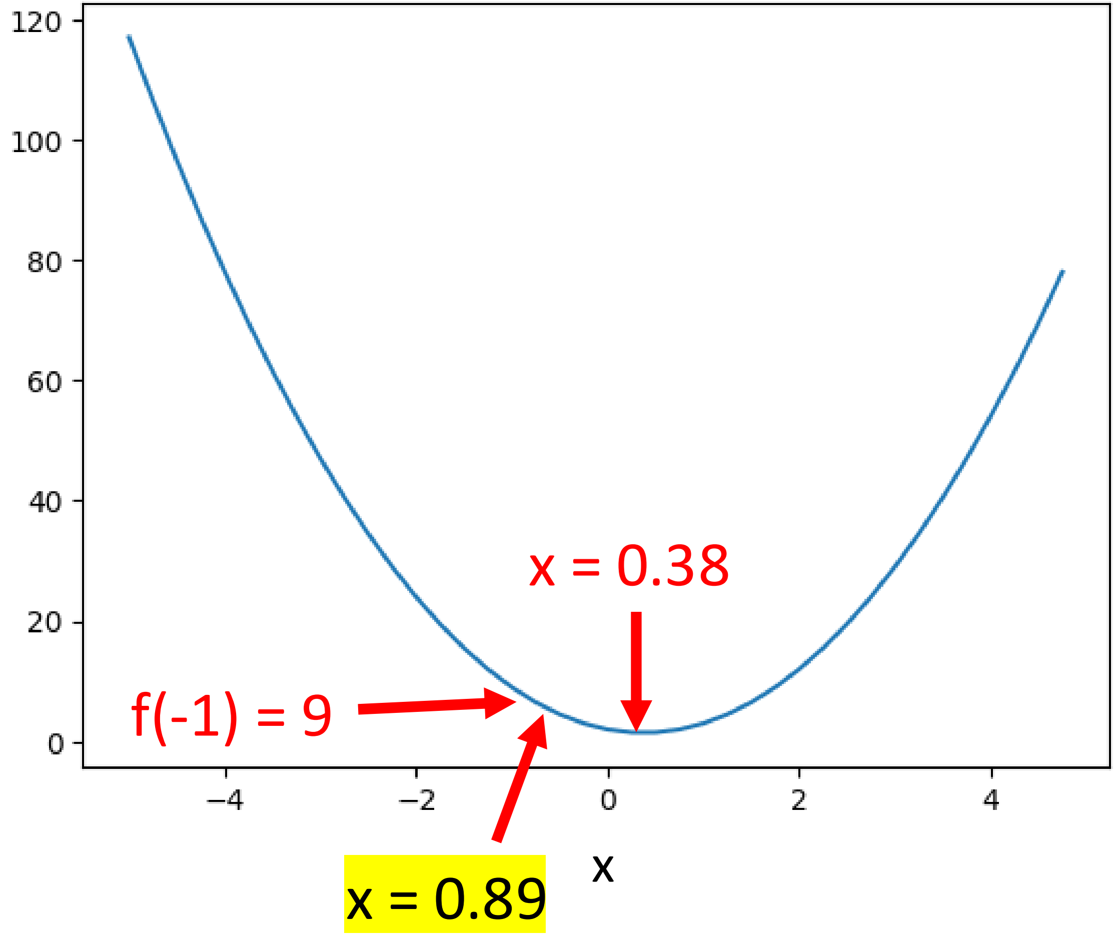
$f(x) = 2 - 3x + 4x^2$

Derivative indicates how a slight change (i.e., h) to every x will change the output:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}$$

e.g., at $x = -1$ and $df/dx = -11$
Approach minimum point by stepping in opposite direction of derivative (e.g., learning rate of 0.01); what should be new value for x?

$-1 - (0.01 * -11) = -0.89$



# Reviewing Fundamentals: Toy Example 2

f(**x**) = a + bc; where **x** is a vector with all parameters

Gradient indicates how a slight change (i.e., h) to any **input** will change the output:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(\mathbf{x} + h) - f(\mathbf{x})}{h}$$

What is the value of the function when a = 1, b = -2, and c = 3?

If we only change *a* with a slightly positive *h*, will f(**x**+h) be greater than or less than -5?

By how much?

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e.g., at a = 1, b = -2, and c = 3 when only changing the value of a

- for h = 0.01: f(**x**+h) = (1+.01) + -2*3 = -4.99, slope = 1.000
- for h = 0.001: f(**x**+h) = (1+.001) + -2*3 = -4.999, slope = 1.000

As h gets closer to 0, slope approaches 1

Mathematically, df/da = 1

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$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(\mathbf{x} + h) - f(\mathbf{x})}{h}$$

Recall the value of the function when a = 1, b = -2, and c = 3 is -5

If we change only *b* with a slightly positive *h*, will f(**x**+h) be greater than or less than -5?

By how much?

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f(**x**) = a + bc; where **x** is a vector with all parameters

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$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(\mathbf{x} + h) - f(\mathbf{x})}{h}$$

e.g., at a = 1, b = -2, and c = 3 when only changing the value of b (rounding to 3 decimals)
- for h = 0.01: f(**x**+h) = 1 + (-2 +.01)*3 = -4.970, slope = 3.000
- for h = 0.001: f(**x**+h) = 1 + (-2 +.001)*3 = -4.997, slope = 3.000

As h gets closer to 0, slope approaches 3

Mathematically, df/db = c (which we set to 3)

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Recall the value of the function when a = 1, b = -2, and c = 3 is -5

If we change only *c* with a slightly positive *h*, will f(**x**+h) be greater than or less than -5?

By how much?

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e.g., at  a = 1, b = -2, and c = 3 when only changing the value of c (rounding to 3 decimals)
- for h = 0.01: f(**x**+h) = 1 + -2*(3 +.01) = -5. 02, slope = -2.000
- for h = 0.001: f(**x**+h) = 1 + -2*(3 +.001) = -5.002, slope = -2.000

As h gets closer to 0, slope approaches -2

Mathematically, df/dc = b (which we set to -2)

# Reviewing Fundamentals: Toy Example 2

f(**x**) = a + bc; where **x** is a vector with all parameters

Gradient indicates how a slight change (i.e., h) to any **input** will change the output:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(\mathbf{x} + h) - f(\mathbf{x})}{h}$$

In summary, gradient for any input can be computed with these equations:

df/da = 1; df/db = c; df/dc =b

Can minimize objective function by stepping (e.g., learning rate of 0.01) in opposite direction of derivative; e.g., at a = 1, b = -2, and c = 3

a - 0.01*1 = 0.99; b - 0.01*c = -2.03; c - 0.01*b = 3.02 f(x) = -5.14

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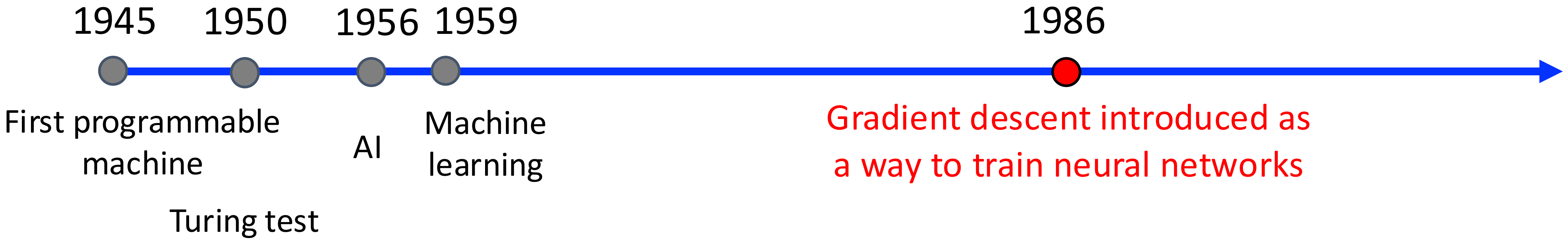
- Gradient descent: how neural networks learn
- Mathematical foundation of gradient descent: derivatives
- Applying gradient descent to train neural networks
- Training example

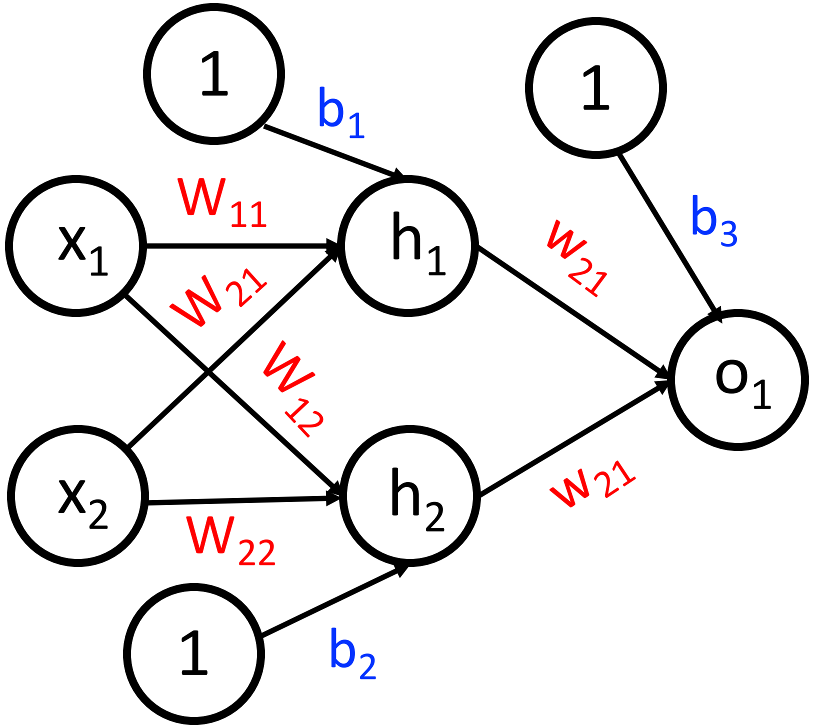
# Training Neural Networks

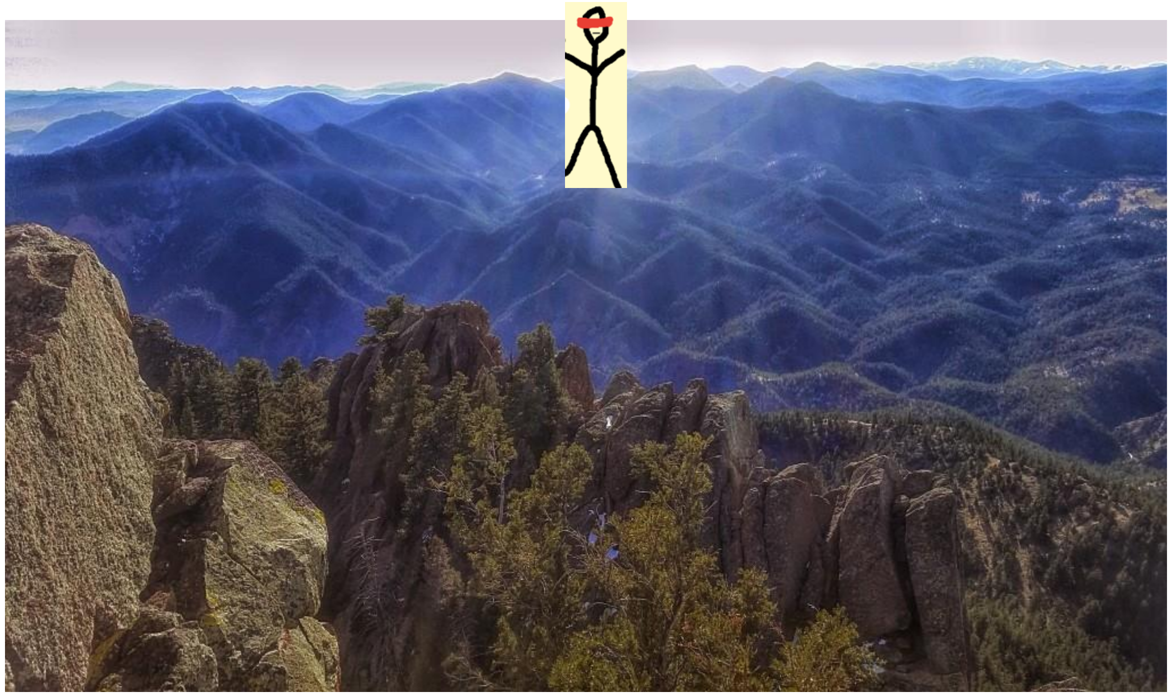



D. Rulhart, G. Hinton, and R. Williams, Learning Internal Representations by Error Propagation, 1986.

# Training Neural Networks

- Descend the loss function with gradient descent; e.g., (weights, biases)



# Gradient Descent: Implementation

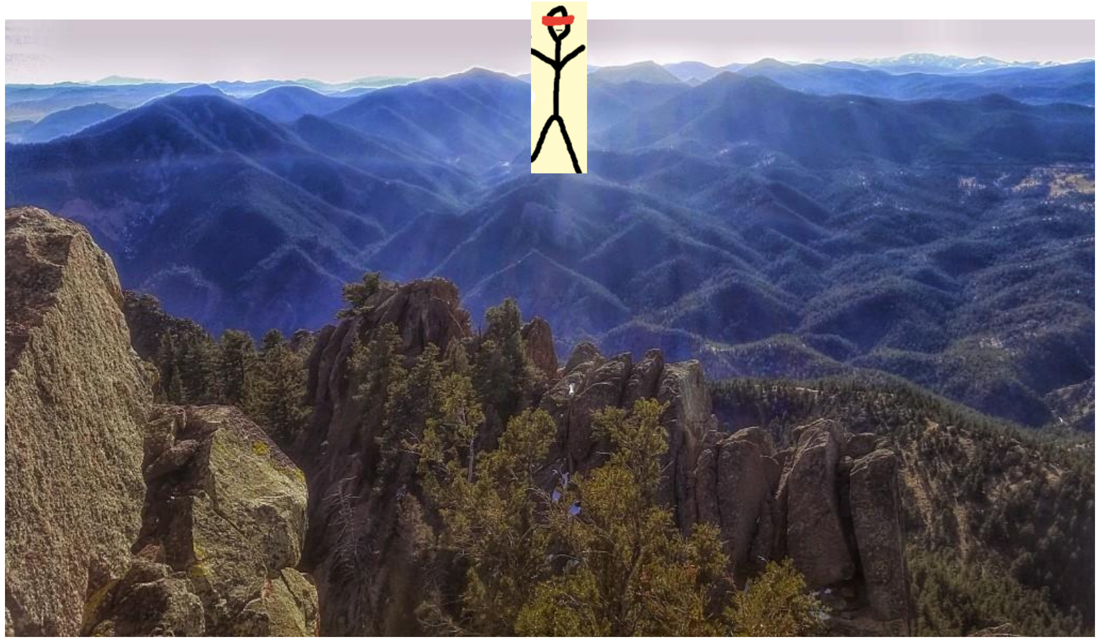
- Repeat until stopping criterion met:

1. Propagate training data through model to make predictions
2. Measure error of the model's predictions on training data using a loss function
3. $\boxed{\text{Calculate gradients to determine how each model parameter contributed to model error}}$
4. Update each parameter using calculated gradients

Breakthrough in 1986: backpropagation used to compute the gradient

# Breakthrough: Backpropagation

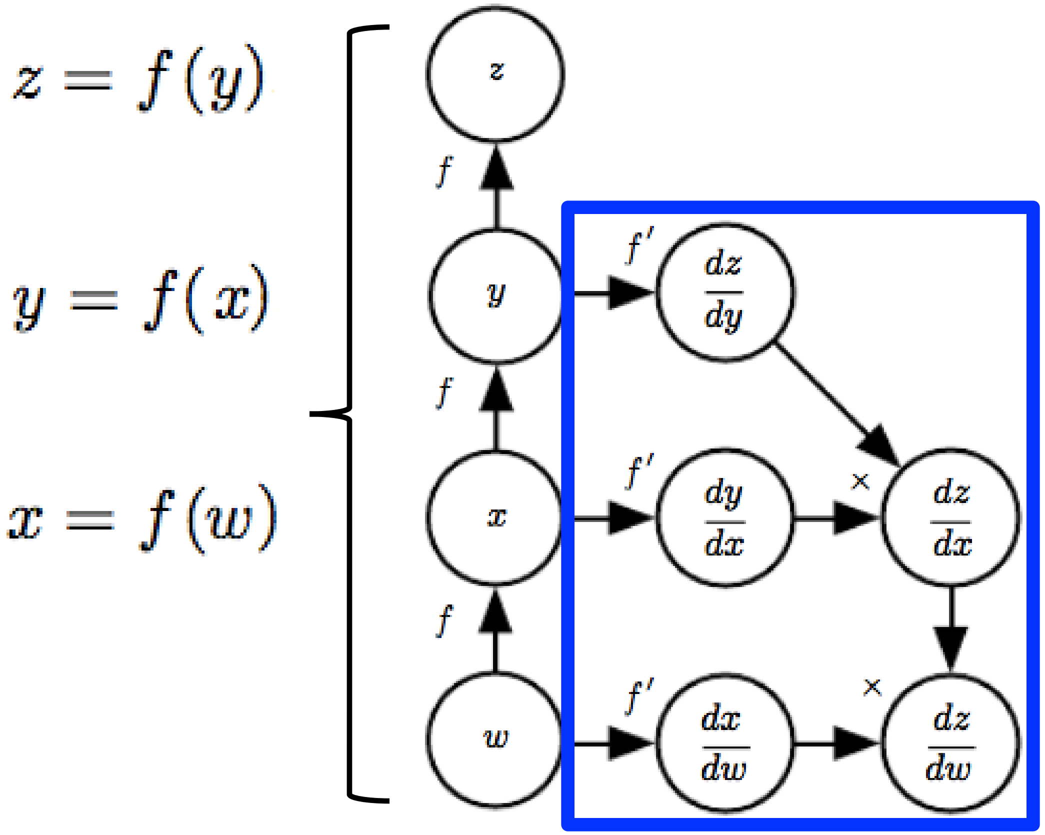
- **Key observation**: loss function is a mathematical function, that takes as input training data and model parameters (e.g., weights, biases)



Using backpropagation, we can know how to tweak all of a function's variables to minimize an error/loss function

# Breakthrough: Backpropagation

- **Key observation**: loss function is a mathematical function connected in a chain
- **What we want:** partial derivatives for function variables to inform how to minimize the loss function
- **Backpropagation**: achieves this with chain rule of calculus, which relies only on local derivatives; e.g.,




Indicates how much a small change to each variable changes the function's output

Ian Goodfellow, Yoshua Bengio, and Aaron Courville; Deep Learning, 2016.

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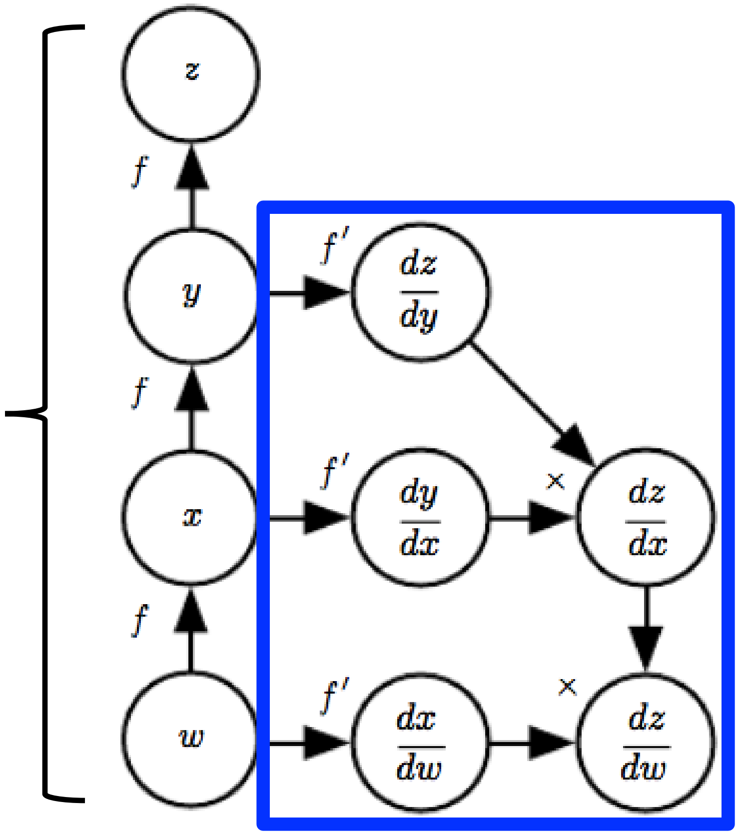
$$z = f(y)$$

$$y = f(x)$$

$$x = f(w)$$




What is *dz/dx*? $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$ (how changes in x will change the output, z)

Ian Goodfellow, Yoshua Bengio, and Aaron Courville; Deep Learning, 2016.

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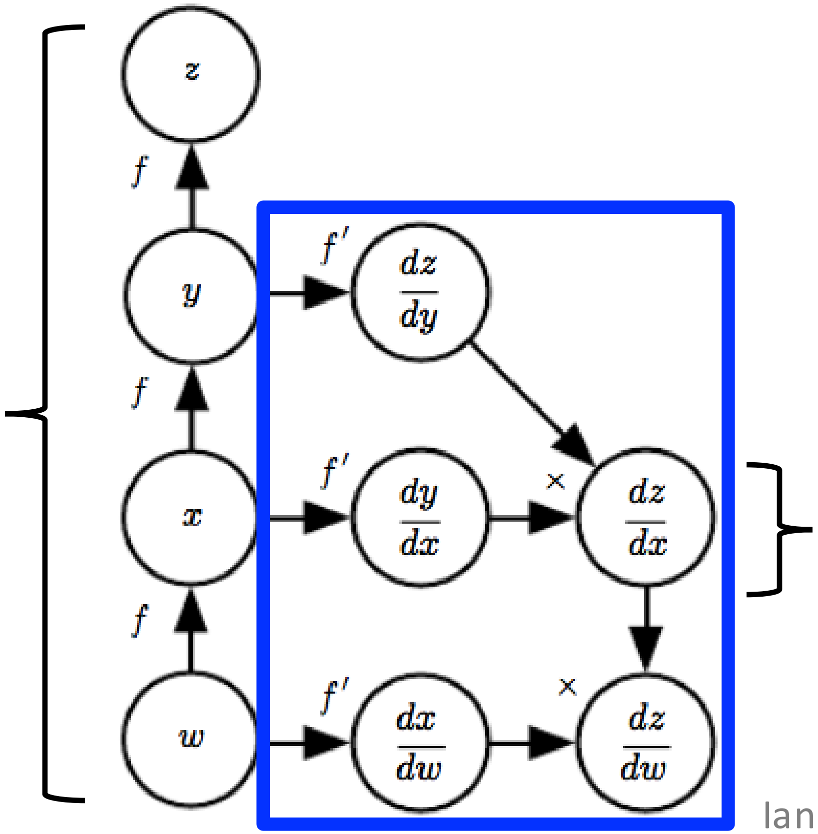


What is *dz/dx*?

Intuitive example: how much faster is my husband compared to my daughter? (*dz/dx*)

*dz/dy:* husband is 2x faster than son

*dy/dx:* son is 3x faster than daughter

Ian Goodfellow, Yoshua Bengio, and Aaron Courville; Deep Learning, 2016.

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1. Propagate training data through model to make predictions
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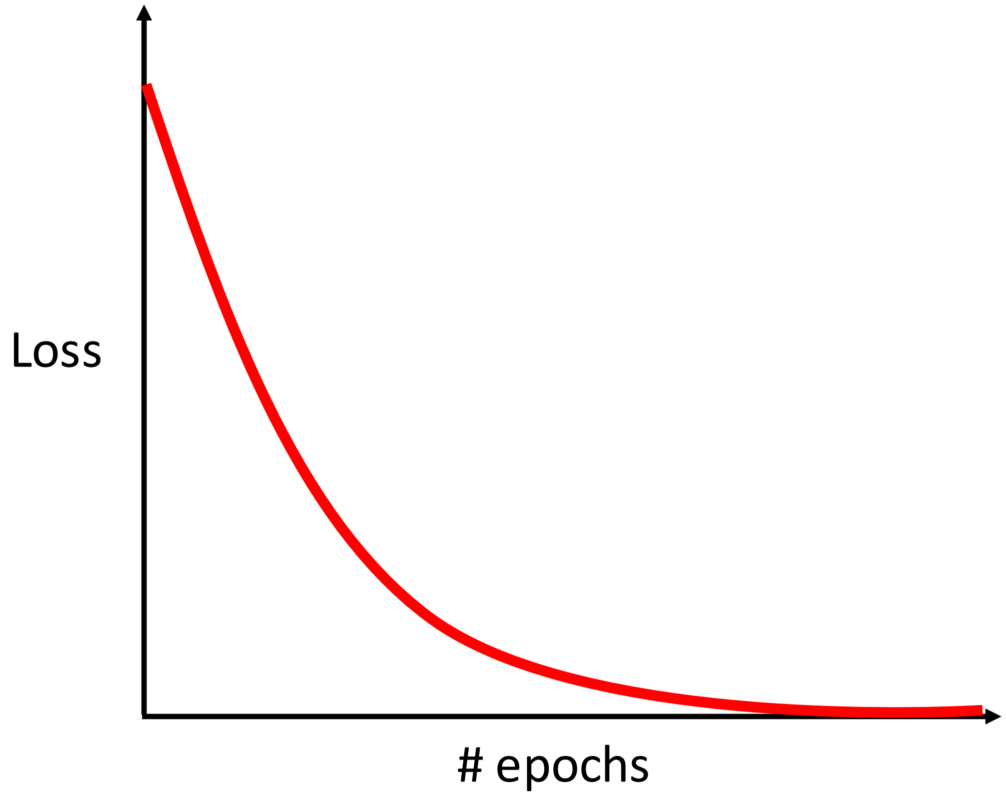
3. Calculate gradients to determine how each model parameter contributed to model error
4. Update each parameter using calculated gradients

We babysit this step to see how learning is going

# Loss Curves Signal How Well Training is Going

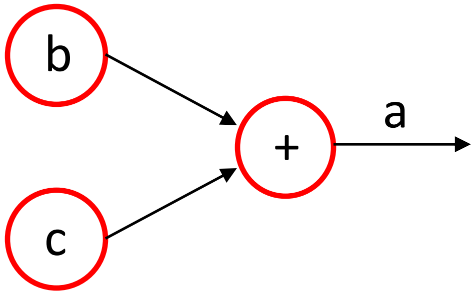
What we want to
see when training:

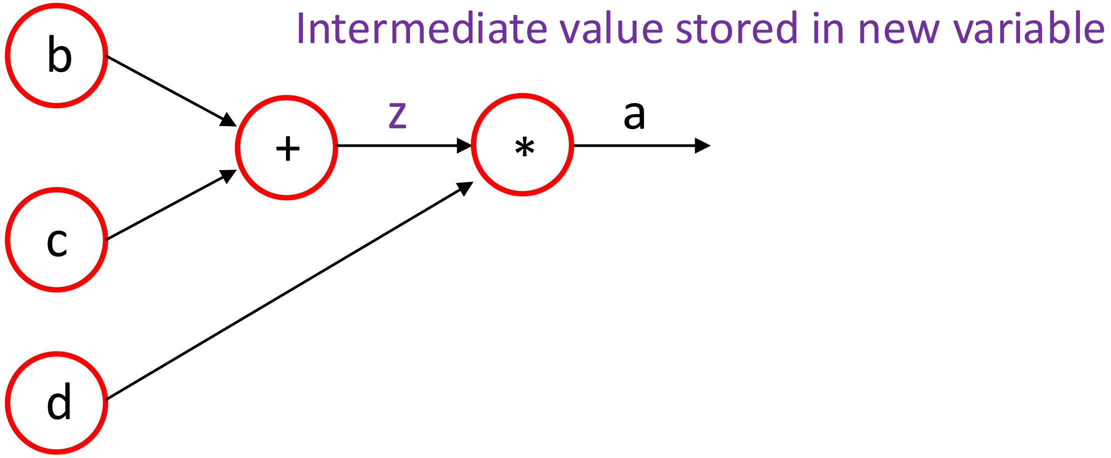


# Abstraction for Training Neural Networks: Computational Graphs

- A directed graph expressing a mathematical expression
  - Nodes: either variables (e.g., input values) or functions (applied to the in-flowing edges)
  - Edges: values output from functions

- For example: a = b + c OR a = d*(b + c)

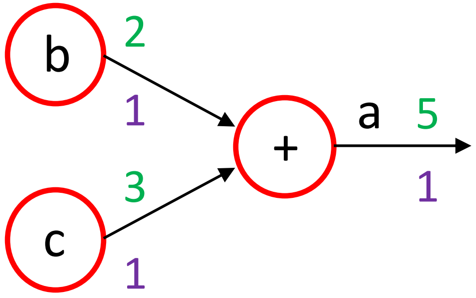


# How to Use a Computational Graph?

- Forward pass: to evaluate an expression, set input variables and then perform all sequential computational steps in the graph
- Backward pass: compute local gradient at each edge and then combine all relevant gradients for each function variable
- For example, using b=2 and c=3:

f(b,c) = b + c




From calculus, we know:

*- df/df = 1*   *- df/db = 1*   *- df/dc = 1*

# How to Use a Computational Graph?

- Forward pass: to evaluate an expression, set input variables and then perform all sequential computational steps in the graph
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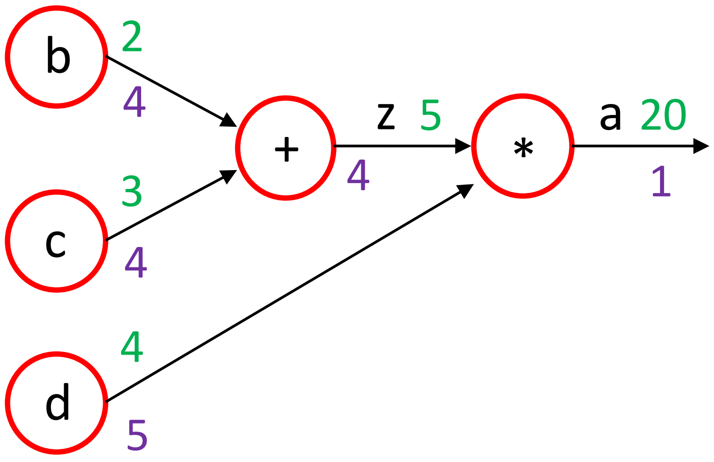
- For example, using b=2, c=3, and d=4:

f(b,c,d) = d*(b + c)




From calculus, we know:

*- df/df = 1* *- df/dz = d* *- df/dd = z*

*- df/db = df/dz * dz/db* *- dz/db = 1*

*- df/dc = df/dz * dz/dc* *- df/dc = 1*

Multiplication passes the other value through and addition passes the incoming gradient

# Motivation for Computational Graphs

- Many deep learning frameworks rely on computational graphs; why?
  - **Efficient representation**: modular, primitive functions can be reused and combined to build complex models
  - **Efficient computation:** derivatives can be calculated based on "local" derivatives of the primitive functions in the graph, which enables them to be computed in parallel and stored for later reuse
  - **Interpretability:** simplifies understanding, debugging, and identifying bottlenecks in the network

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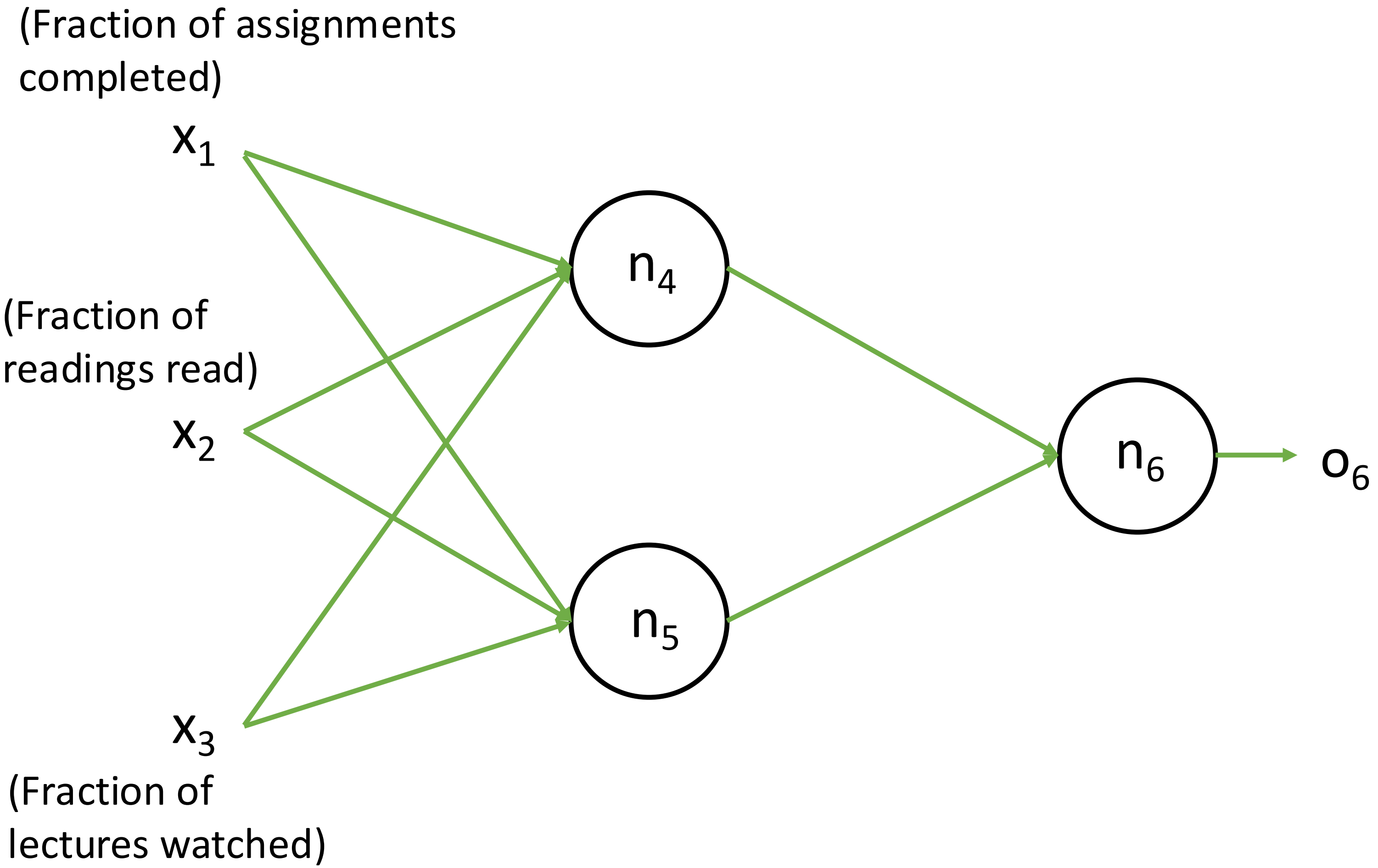
# Example

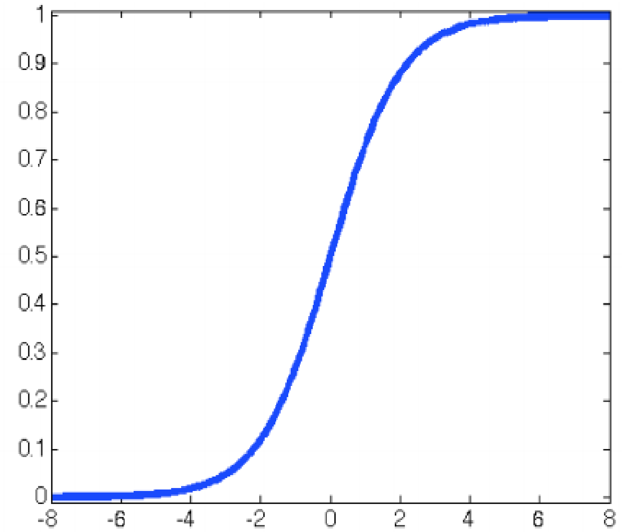
- Binary classification: predict if a student will get a B or better (minimum required for CS graduate student in this course)

- Inputs:
    - Fraction of assignments completed
    - Fraction of readings read
    - Fraction of lectures watched

# Example: Choose Neural Network Architecture



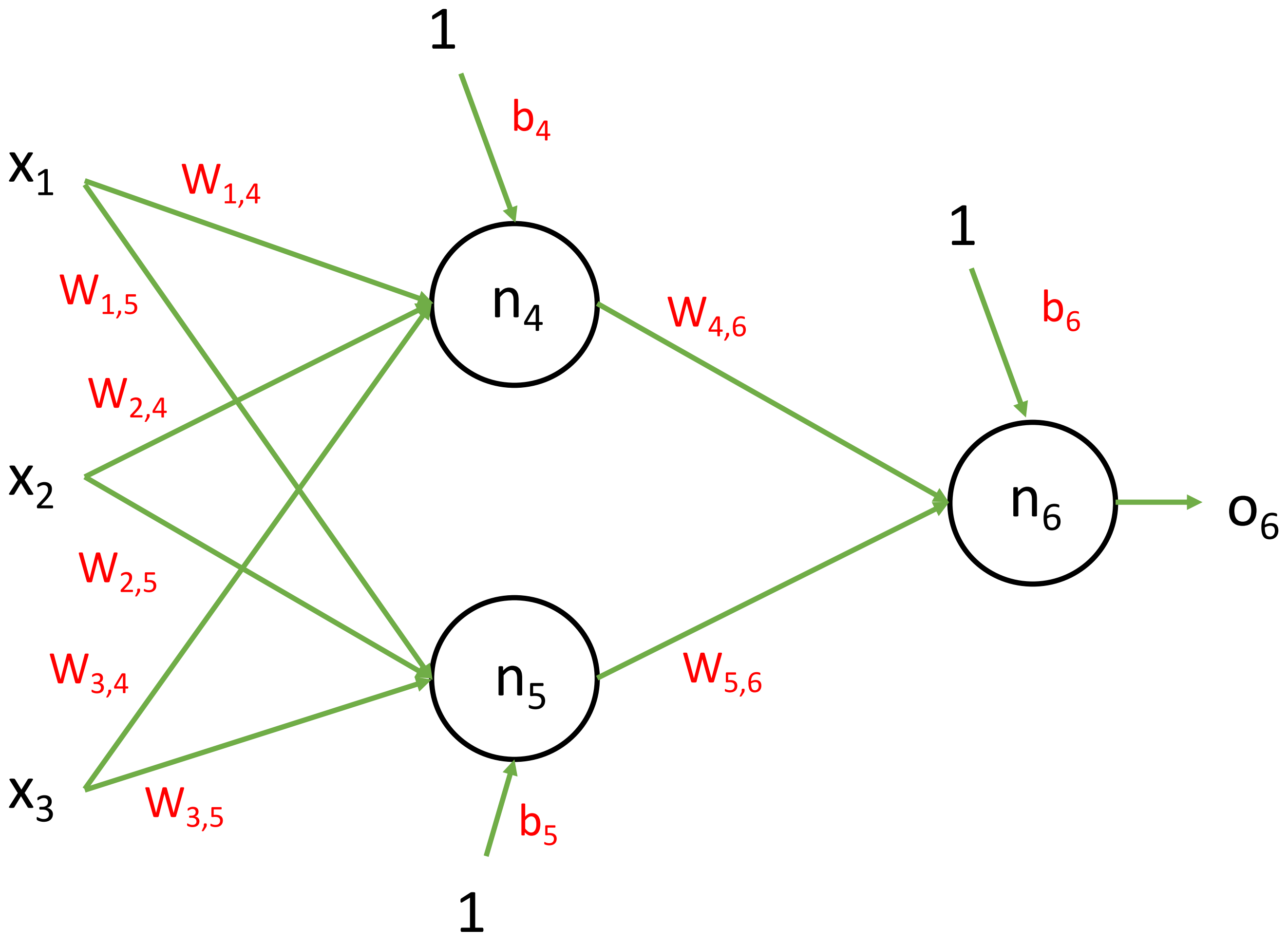

Sigmoid Activation Function




$$\sigma(z) = \frac{1}{1+\exp(-z)}$$

Example from: Jiawei Han and Micheline Kamber; Data Mining.

# Example: Choose Neural Network Architecture




2-layer neural network

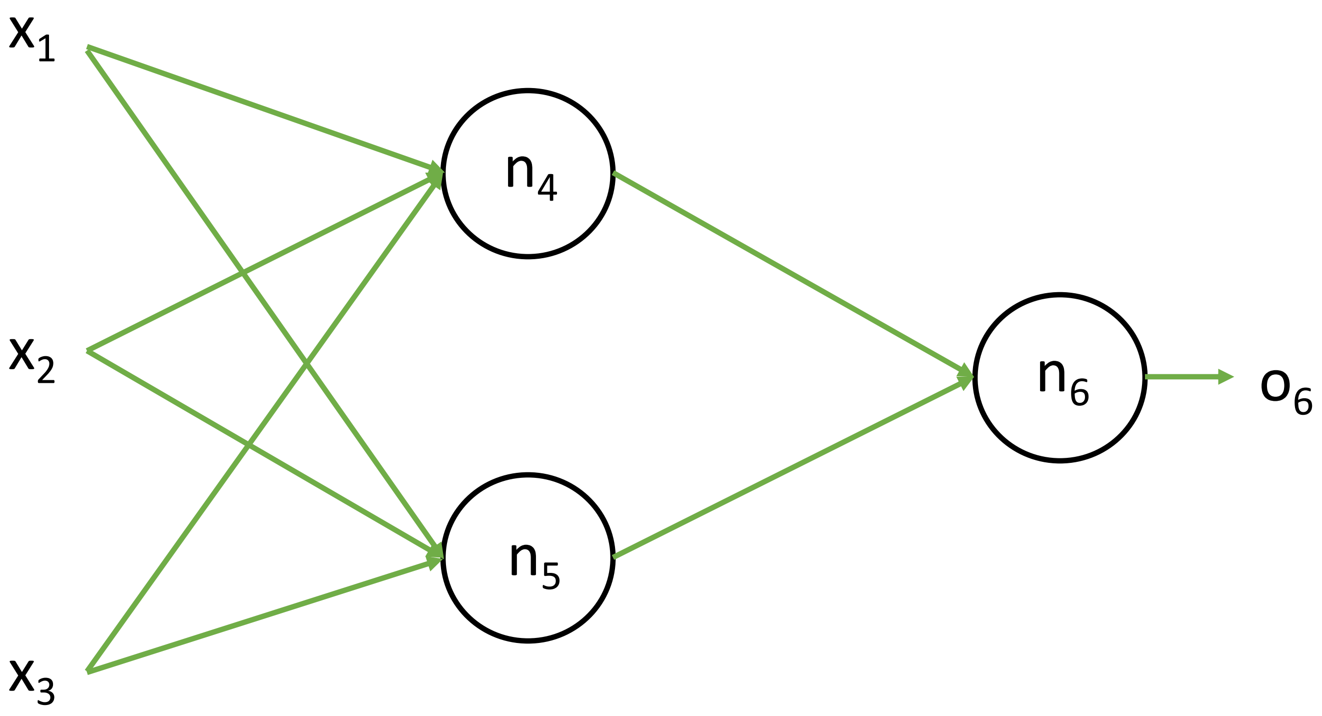
8 weights:

- Input to Hidden Layer 1:
  - 3x2 = 6
- Hidden Layer 1 to Hidden Layer 2:
  - 2x1 = 2

- 3 bias terms

Example from: Jiawei Han and Micheline Kamber; Data Mining.

# Example: Choose Loss Function




Squared Error

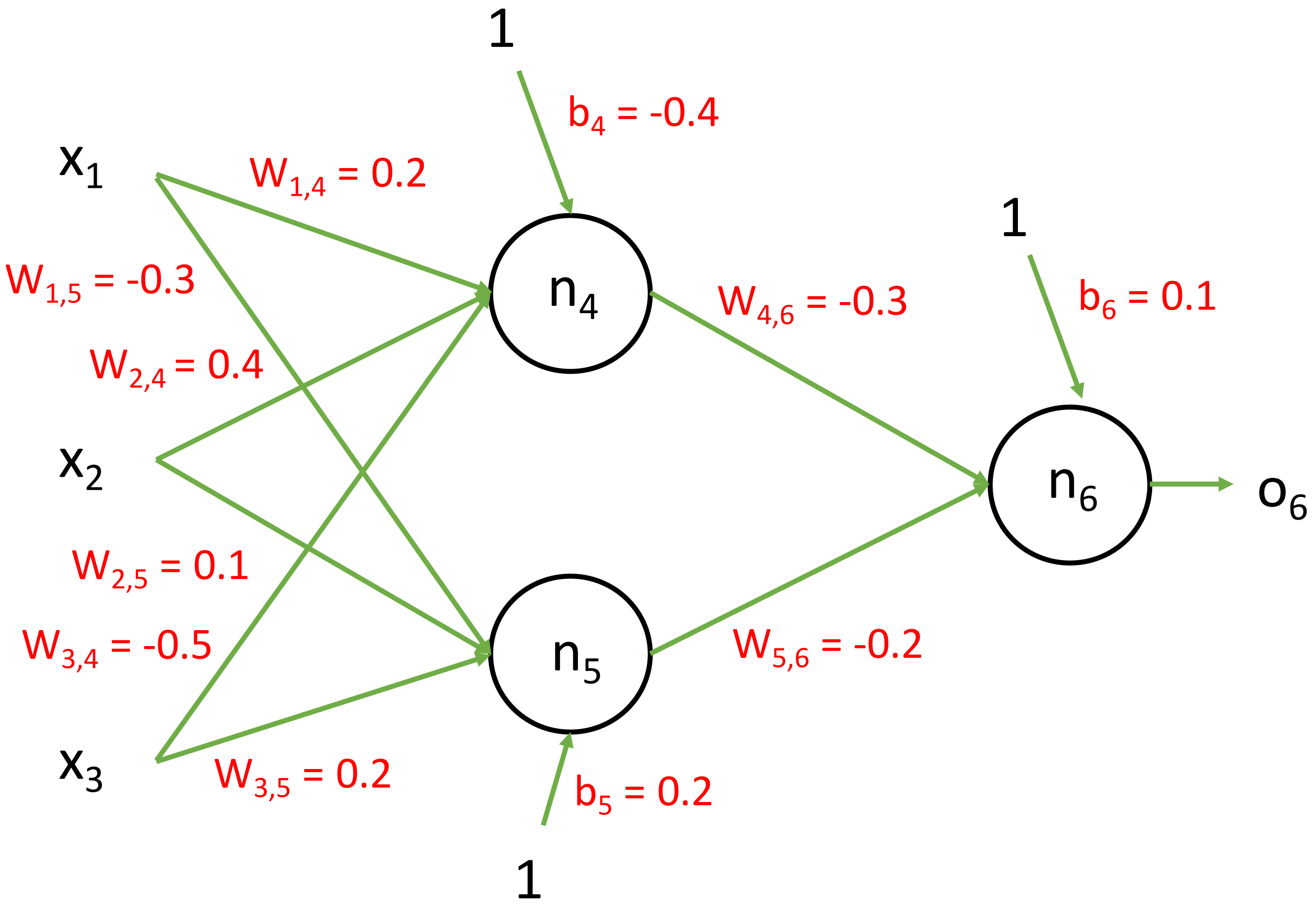
$$(Y_i - \hat{Y}_i)^2$$

True value

Predicted value

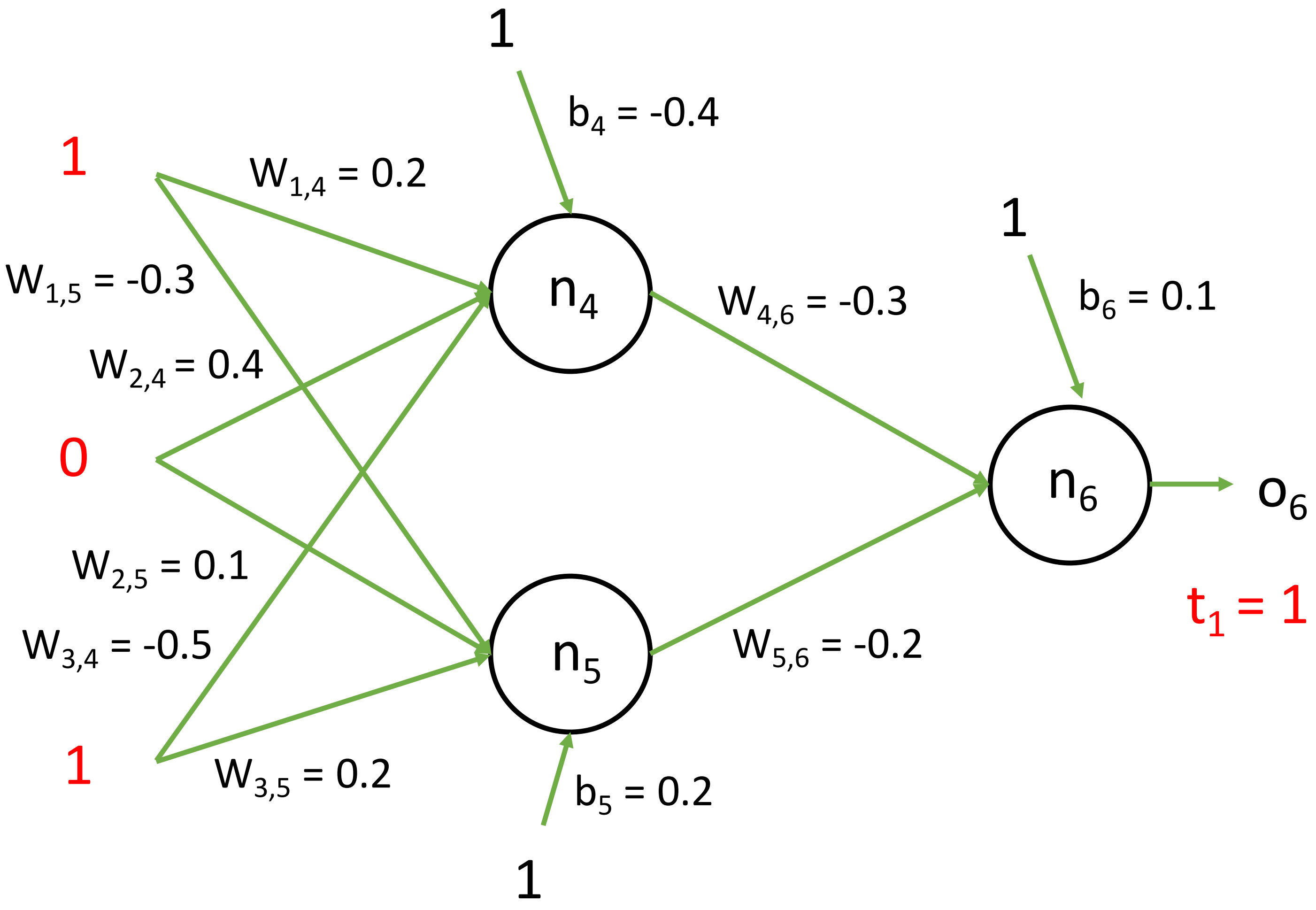
Example from: Jiawei Han and Micheline Kamber; Data Mining.

# Example: Initialize Model Parameters



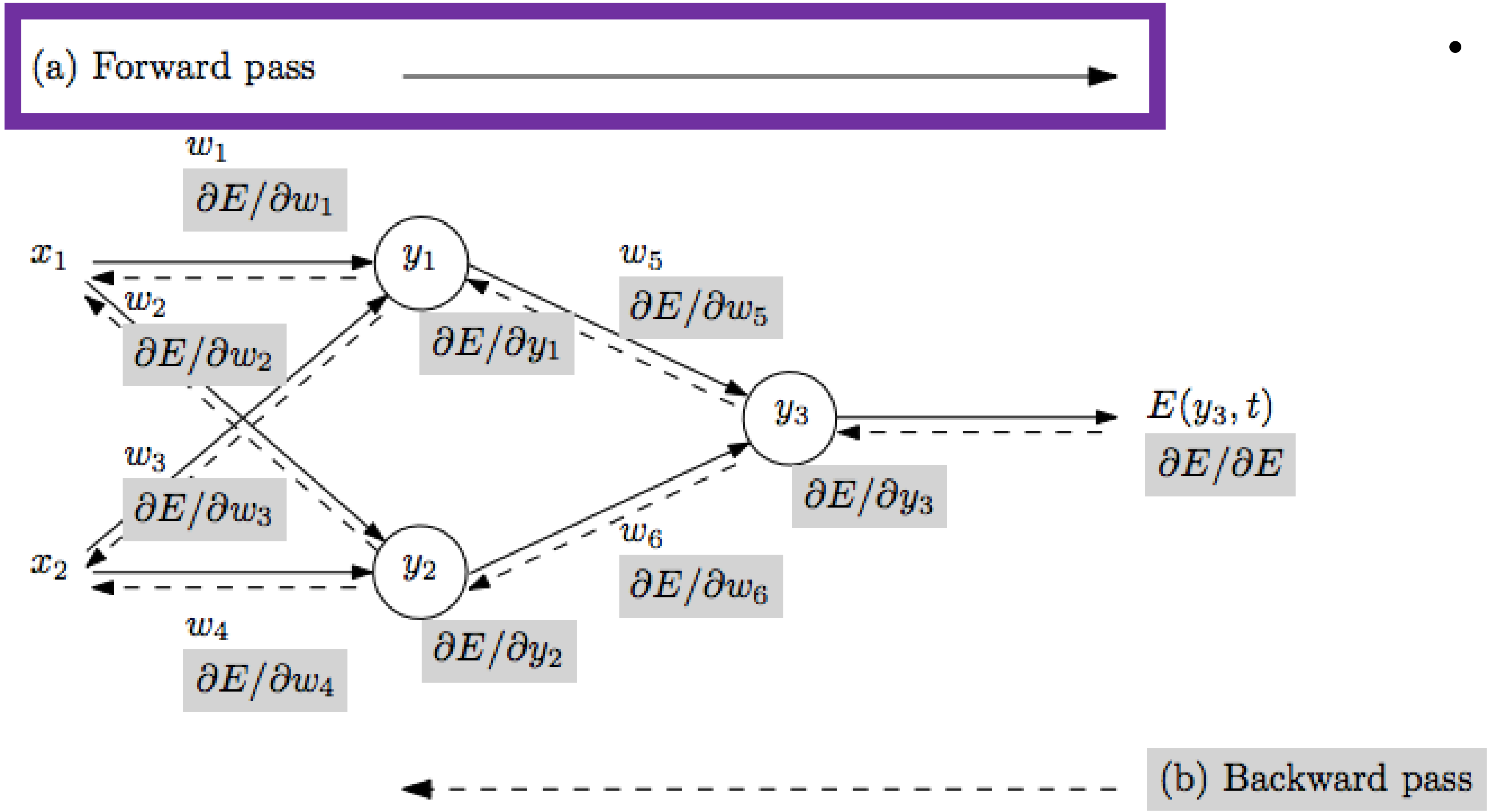

Example from: Jiawei Han and Micheline Kamber; Data Mining.

# Example: Input Training Example




Example from: Jiawei Han and Micheline Kamber; Data Mining.

# Example: Step 1 – Forward Pass




- Repeat until stopping criterion met:
    1. **Forward pass**: propagate training data through model to make predictions

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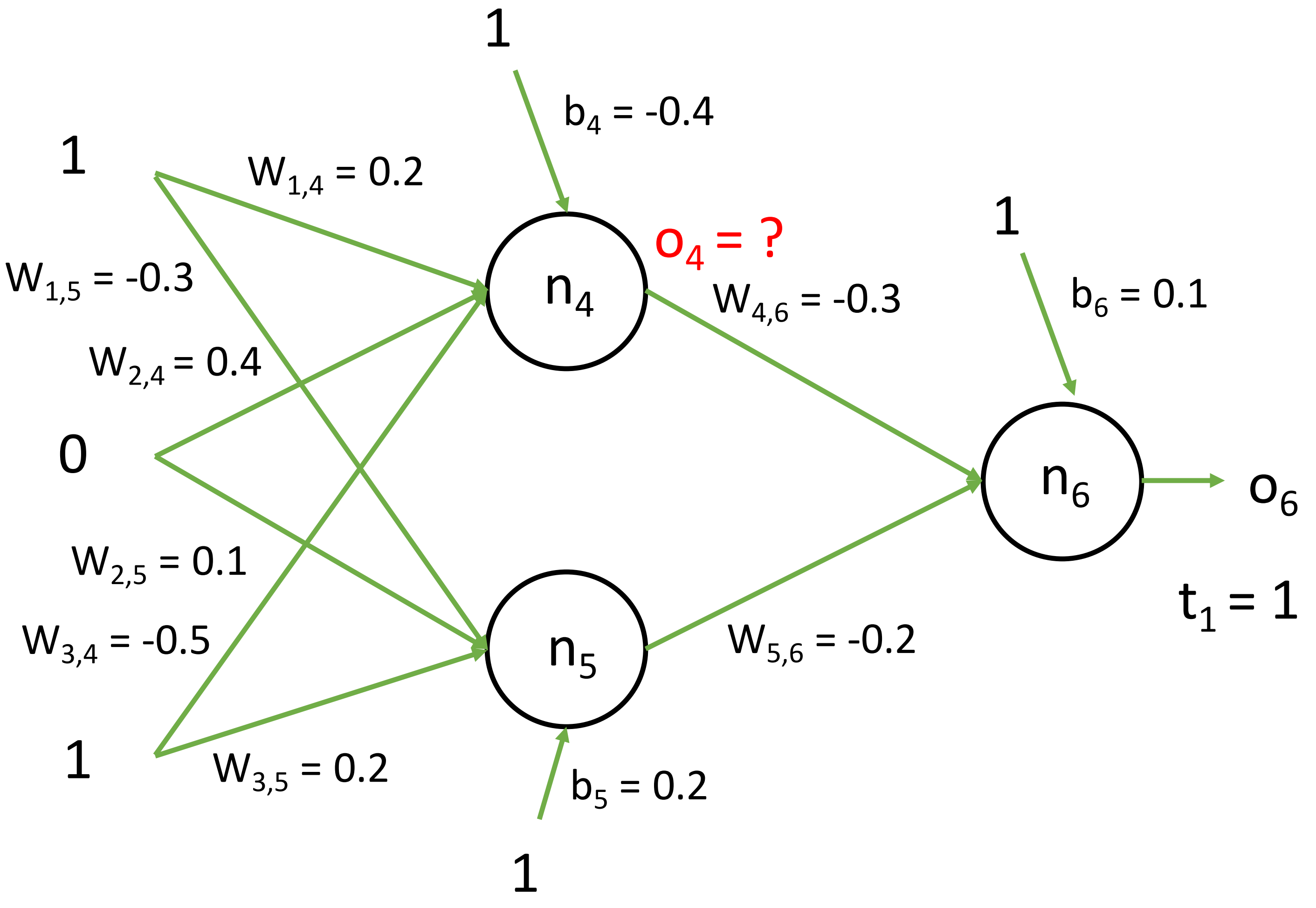
# Example: Step 1 – Forward Pass




Weighted input sum to node 4:

$i_4 = (1 \times 0.2 + 0 \times 0.4 + 1 \times -0.5) - 0.4$

$i_4 = -0.7$

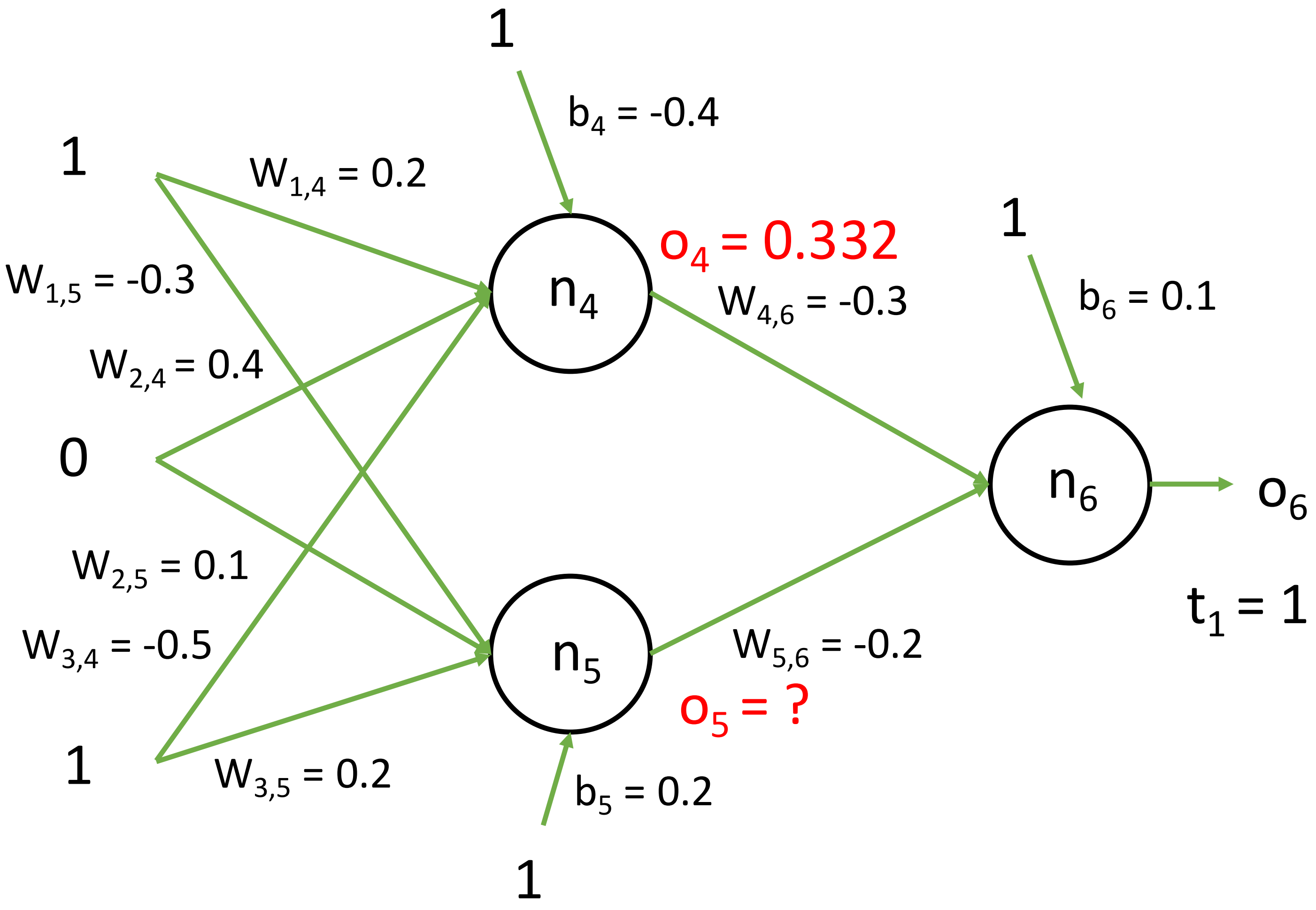
Activation result from node 4:

$o_4 = \textit{sigmoid}(-0.7)$

$o_4 = 1/(1+e^{-(-0.7)})$

$o_4 = 0.332$

# Example: Step 1 – Forward Pass




Weighted input sum to node 5:

$i_5 = (1 \text{ x } -0.3 + 0 \text{ x } 0.1 + 1 \text{ x } 0.2) + 0.2$

$i_5 = 0.1$

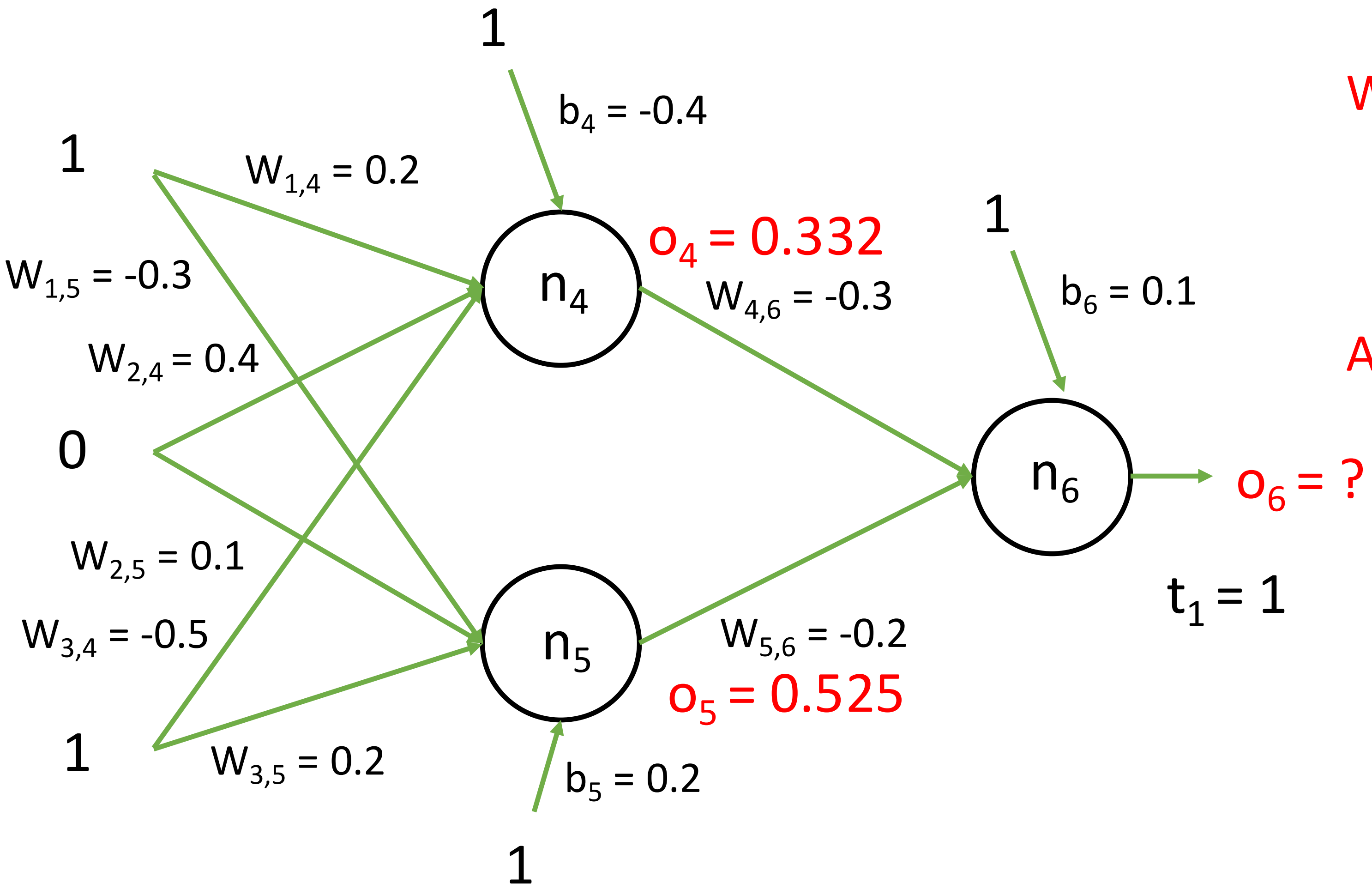
Activation result from node 5:

$o_5 = \textit{sigmoid}(0.1)$

$o_5 = 1/(1+e^{-0.1})$

$o_5 = 0.525$

# Example: Step 1 – Forward Pass




Weighted input sum to node 6:

$i_6 = (0.332 \times -0.3 + 0.525 \times -0.2) + 0.1$

$i_6 = -0.105$

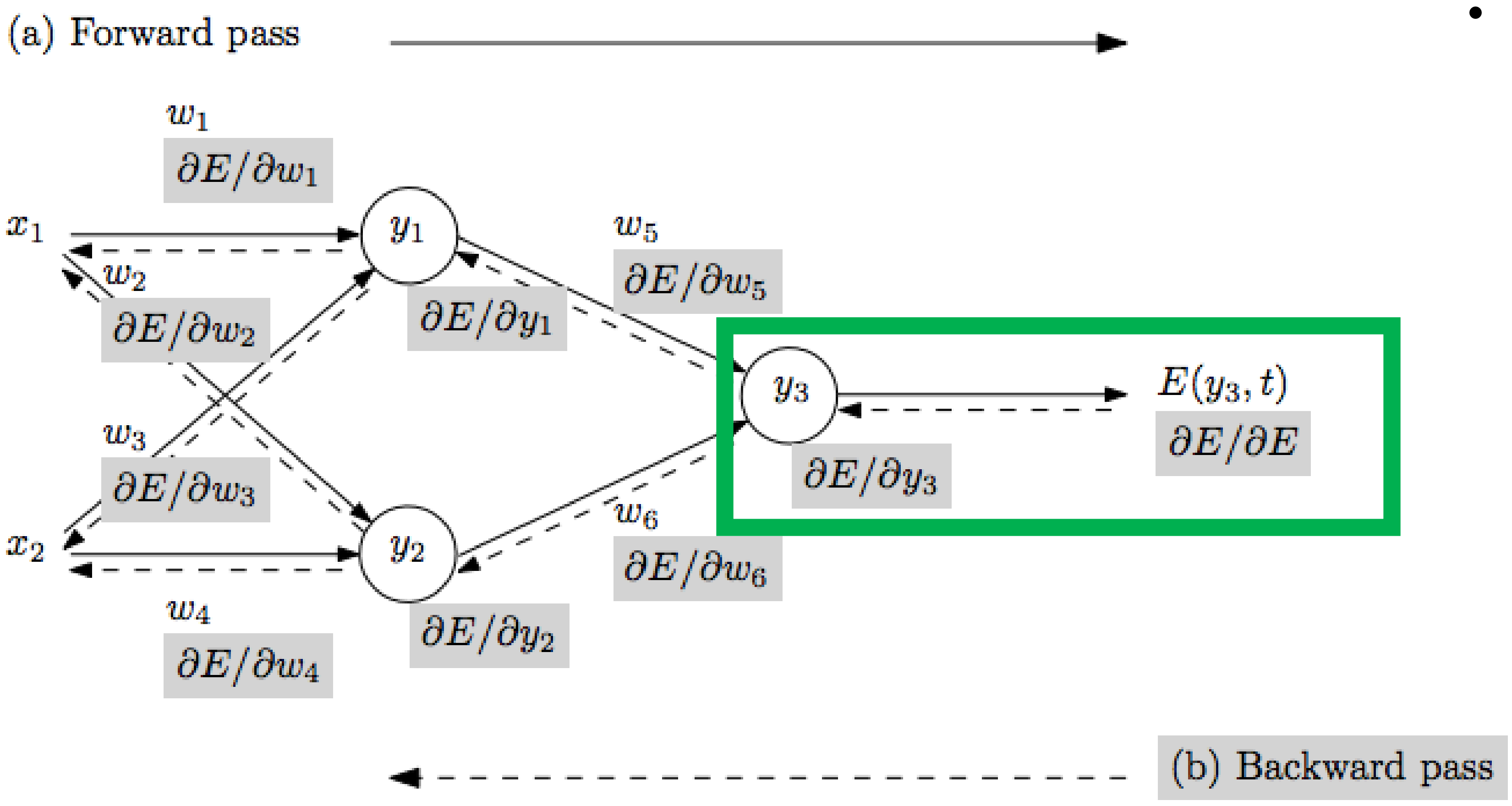
Activation result from node 6:

$o_6 = \textit{sigmoid}(-0.105)$

$o_6 = 1/(1+e^{-(-0.105)})$

$o_6 = 0.474$

# Example: Step 2 – Calculate Error (Loss)




- Repeat until stopping criterion met:
  1. **Forward pass**: propagate training data through model to make predictions
  2. **Error quantification**: measure error of the model's predictions on training data using a loss function

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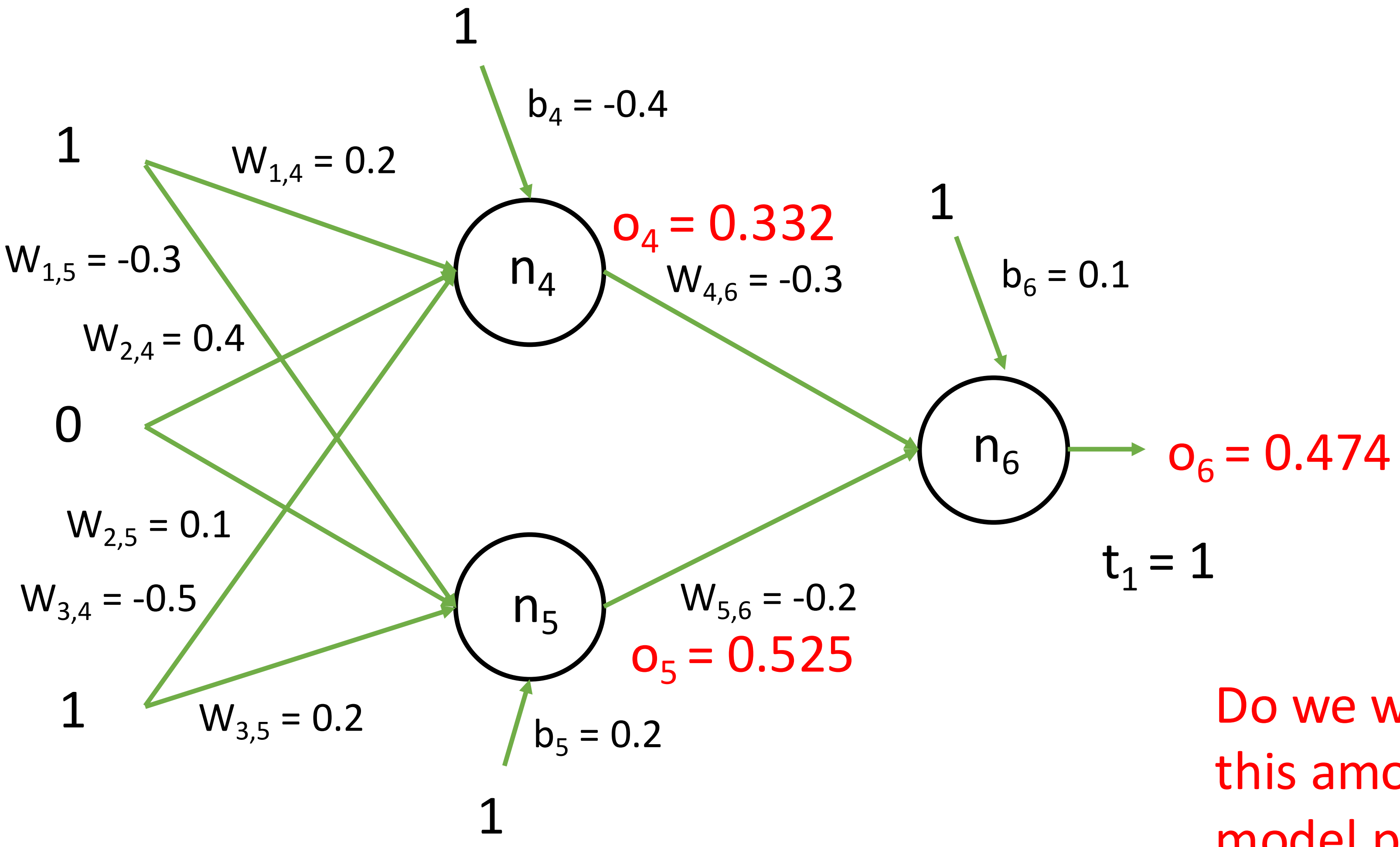
# Example: Step 2 – Calculate Error (Loss)



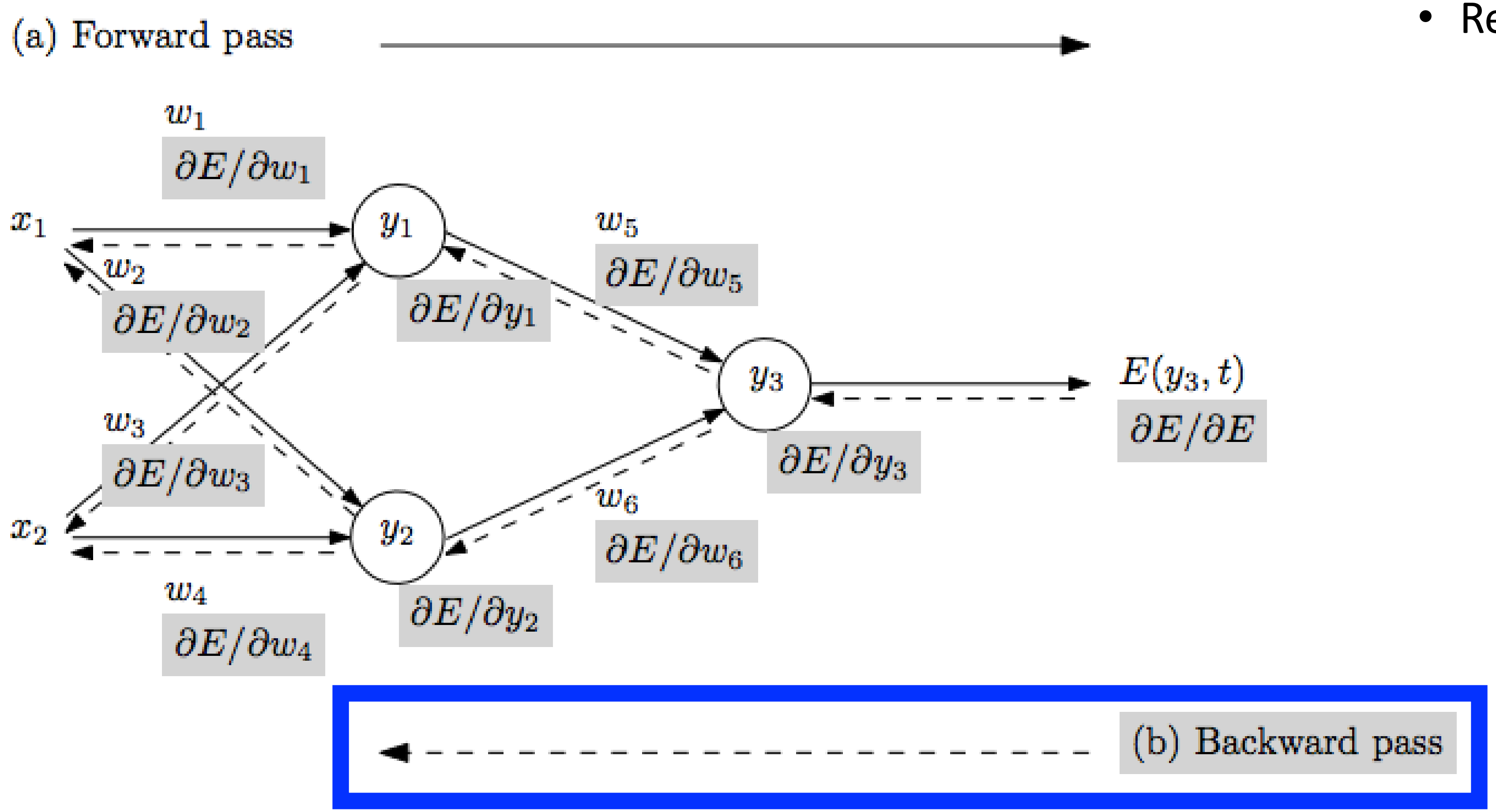

Squared Error

$= (1-0.474)^2$

$= 0.277$

Do we want to increase or reduce this amount when modifying the model parameters during training?

# Example: Step 3 – Backward Pass




- Repeat until stopping criterion met:
  1. **Forward pass**: propagate training data through model to make predictions
  2. **Error quantification**: measure error of the model's predictions on training data using a loss function
  3. **Backward pass**: calculate gradients to determine how each model parameter contributed to model error

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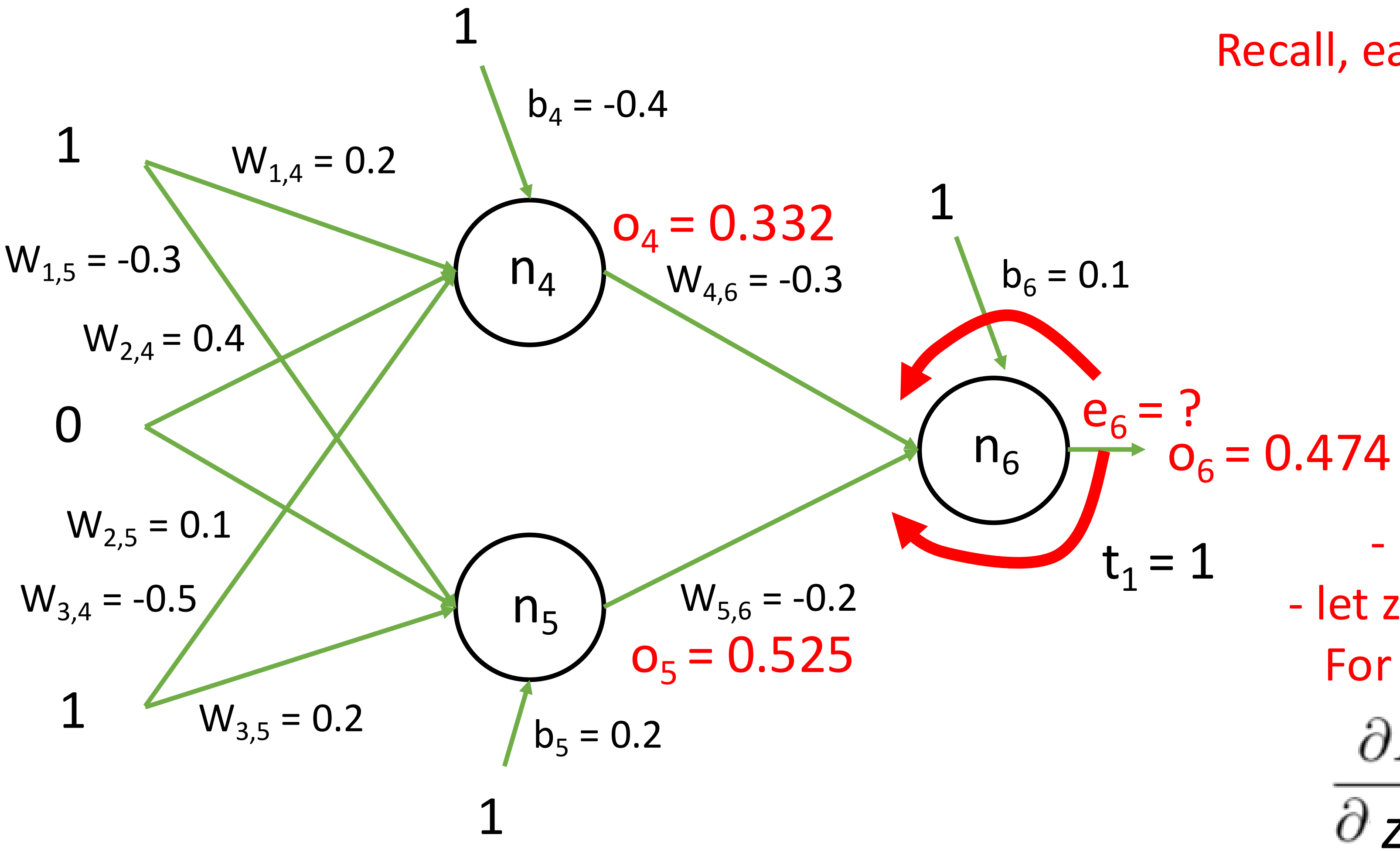
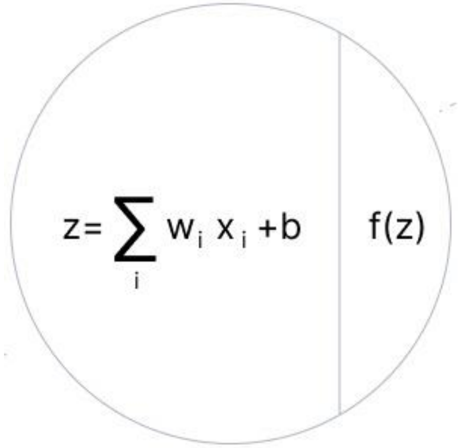
# Example: Step 3 – Backward Pass




Recall, each node contains 2 functions:




- let $o_k$ represent the activation
- let $z_k$ represent the weighted sum

For node 6, we will first compute:

$$\frac{\partial E}{\partial z_6} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6}$$

Image source: https://www.v7labs.com/blog/neural-networks-activation-functions

# Example: Step 3 – Backward Pass

Loss function:

constant value

$$\frac{\partial E}{\partial o_6} = \frac{\partial}{\partial o_6}(t_k - o_6)^2 = \boxed{-2(t_6 - o_6)}$$

Activation function (sigmoid):

$$\frac{\partial o_6}{\partial z_6} = \frac{\partial}{\partial z_6}\frac{1}{1+e^{-z_6}} = (1-\sigma(z_6))\,\sigma(z_6)$$

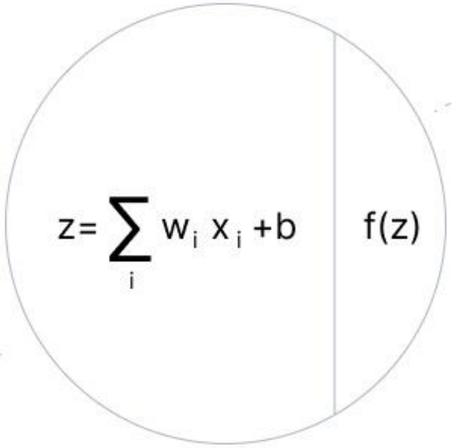
$$= \boxed{(1 - o_6)\,o_6}$$

Putting them together:

$$\frac{\partial E}{\partial z_6} = \boxed{-\cancel{2}(t_6 - o_6)}\ \boxed{(1 - o_6)\,o_6}$$

Can drop constant (does not affect learning)

Recall, each node contains 2 functions:




- let $o_k$ represent the activation
- let $z_k$ represent the weighted sum

For node 6, we will first compute:

$$\boxed{\frac{\partial E}{\partial z_6}} = \boxed{\frac{\partial E}{\partial o_6}} * \boxed{\frac{\partial o_6}{\partial z_6}}$$

Image source: https://www.v7labs.com/blog/neural-networks-activation-functions

# Example: Step 3 – Backward Pass

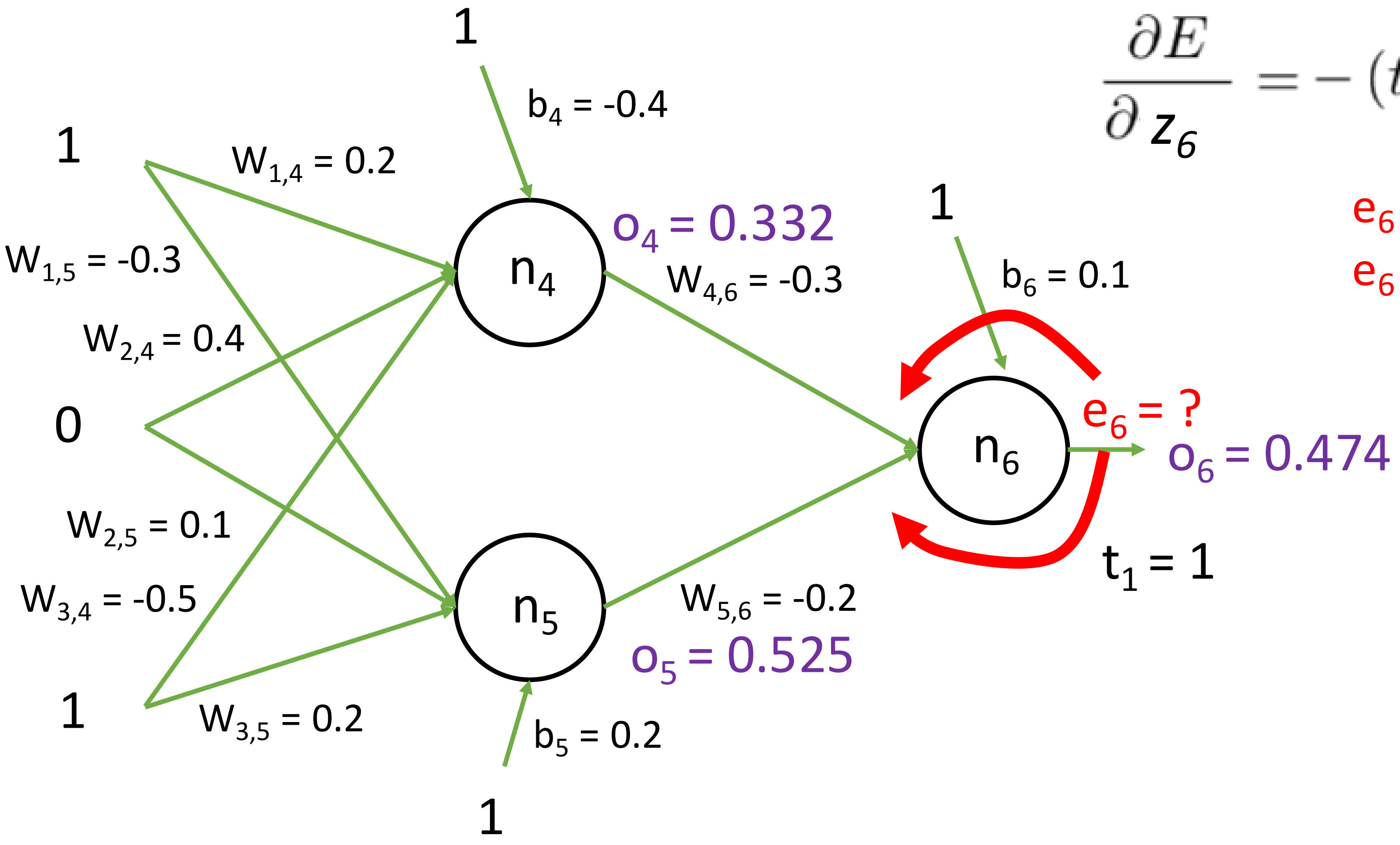
Key Observation: Can compute this because both the activation function and loss function are **differentiable**!!!

$$\frac{\partial E}{\partial z_6} = \boxed{-\cancel{2}(t_6 - o_6)}\ \boxed{(1 - o_6)\ )\ o_6}$$

Can drop constant (does not affect learning)

$$\boxed{\frac{\partial E}{\partial z_6}} = \boxed{\frac{\partial E}{\partial o_6}} * \boxed{\frac{\partial o_6}{\partial z_6}}$$

# Example: Step 3 – Backward Pass

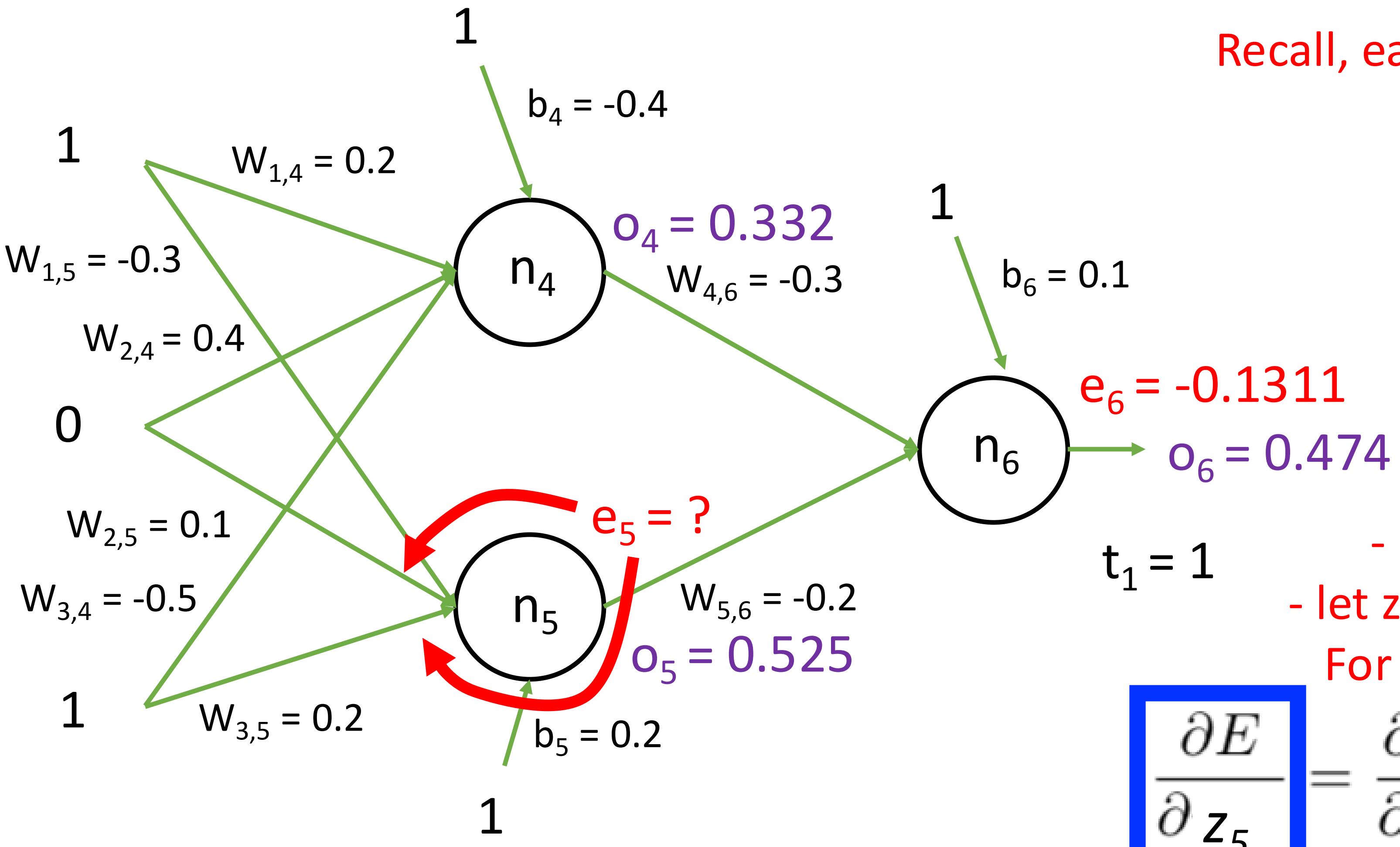



$$\frac{\partial E}{\partial z_6} = -(t_6 - o_6)(1 - o_6)\, o_6$$

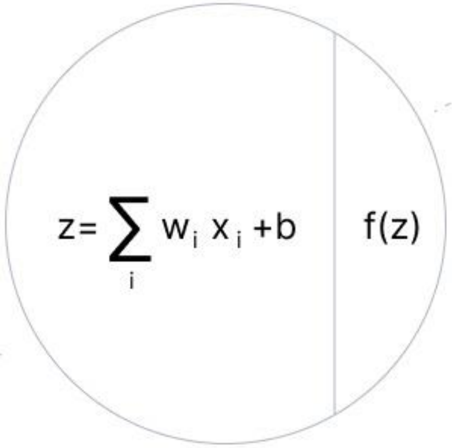
$e_6$ = -(1-0.474)(1-0.474)(0.474)

$e_6$ = -0.1311

# Example: Step 3 – Backward Pass




Recall, each node contains 2 functions:




- let $o_k$ represent the activation
- let $z_k$ represent the weighted sum

For node 5, we will first compute:

$$\boxed{\frac{\partial E}{\partial z_5}} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6} * \boxed{\frac{\partial z_6}{\partial o_5}} * \boxed{\frac{\partial o_5}{\partial z_5}}$$

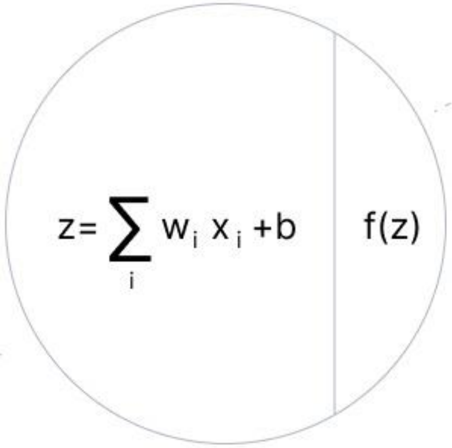
Image source: https://www.v7labs.com/blog/neural-networks-activation-functions

# Example: Step 3 – Backward Pass

Parent node:

$$\frac{\partial z_6}{\partial o_5} = \frac{\partial}{\partial o_5}(o_4 w_{4,6} + o_5 w_{5,6} + b_6) = \boxed{w_{5,6}}$$

Recall, each node contains 2 functions:




Activation function (sigmoid):

$$\frac{\partial o_5}{\partial z_5} = \frac{\partial}{\partial z_5}\frac{1}{1+e^{-z_5}} = (1-\sigma(z_5))\,\sigma(z_5)$$
$$= \boxed{(1-o_5)\,o_5}$$

- let $o_k$ represent the activation
- let $z_k$ represent the weighted sum

For node 5, we will first compute:

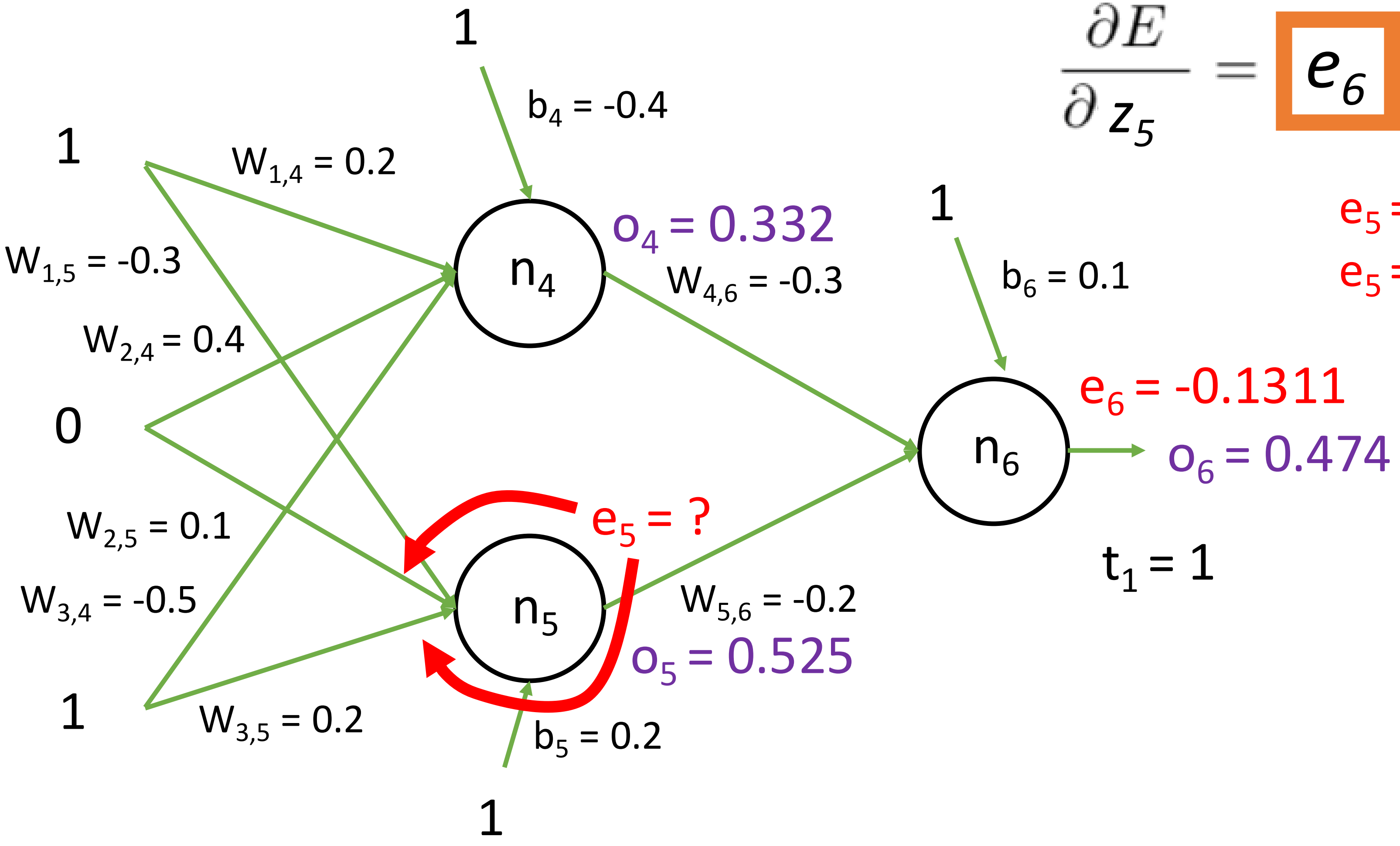
$$\boxed{\frac{\partial E}{\partial z_5}} = \boxed{\frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6}} * \boxed{\frac{\partial z_6}{\partial o_5}} * \boxed{\frac{\partial o_5}{\partial z_5}}$$

Putting them together:

$$\frac{\partial E}{\partial z_5} = \boxed{e_6}\,\boxed{w_{5,6}}\,\boxed{(1-o_5)\,o_5}$$

Image source: https://www.v7labs.com/blog/neural-networks-activation-functions

# Example: Step 3 – Backward Pass

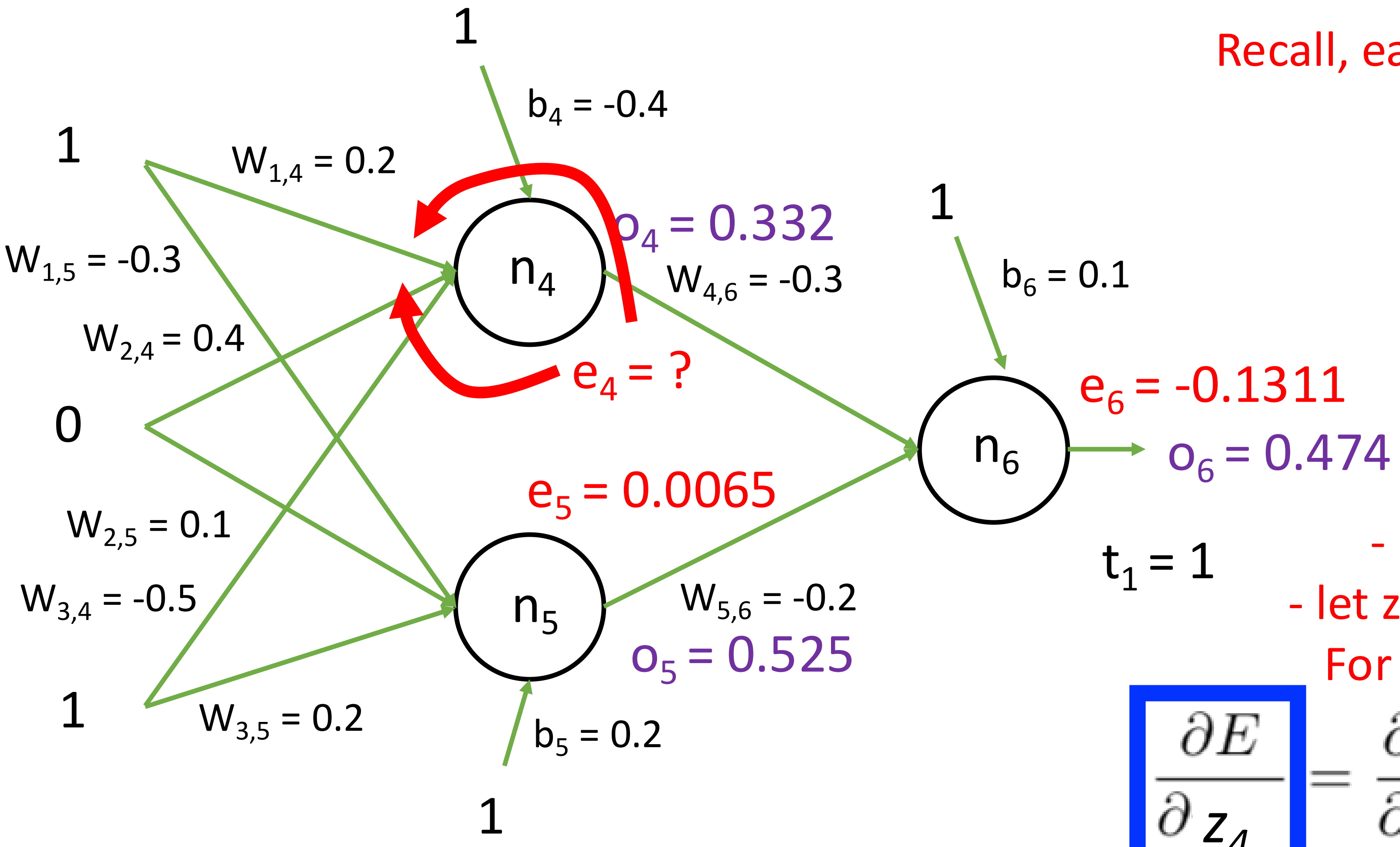



$$\frac{\partial E}{\partial z_5} = \boxed{e_6}\,\boxed{w_{5,6}}\,\boxed{(1 - o_5))\ o_5}$$

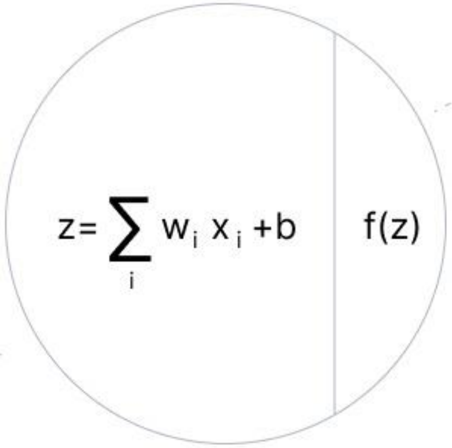
$e_5 = -0.1311 * -0.2 * (1-0.525) * 0.525$

$e_5 = 0.0065$

# Example: Step 3 – Backward Pass




Recall, each node contains 2 functions:




- let $o_k$ represent the activation
- let $z_k$ represent the weighted sum

For node 5, we will first compute:

$$\boxed{\frac{\partial E}{\partial z_4}} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6} * \boxed{\frac{\partial z_6}{\partial o_4}} * \boxed{\frac{\partial o_4}{\partial z_4}}$$

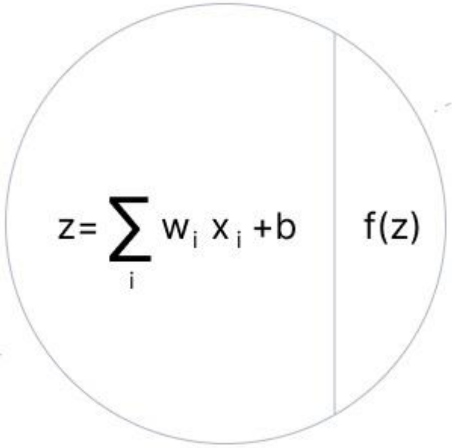
Image source: https://www.v7labs.com/blog/neural-networks-activation-functions

# Example: Step 3 – Backward Pass

Parent node:

$$\frac{\partial z_6}{\partial o_4} = \frac{\partial}{\partial o_4}\left(o_4\, w_{4,6} + o_5\, w_{5,6} + b_6\right) = \boxed{w_{4,6}}$$

Recall, each node contains 2 functions:




Activation function (sigmoid):

$$\frac{\partial o_4}{\partial z_4} = \frac{\partial}{\partial z_4}\,\frac{1}{1+e^{-z_4}} = (1-\sigma(z_4))\,\sigma(z_4)$$

$$= \boxed{(1 - o_4\;))\; o_4}$$

- let $o_k$ represent the activation
- let $z_k$ represent the weighted sum

Putting them together:

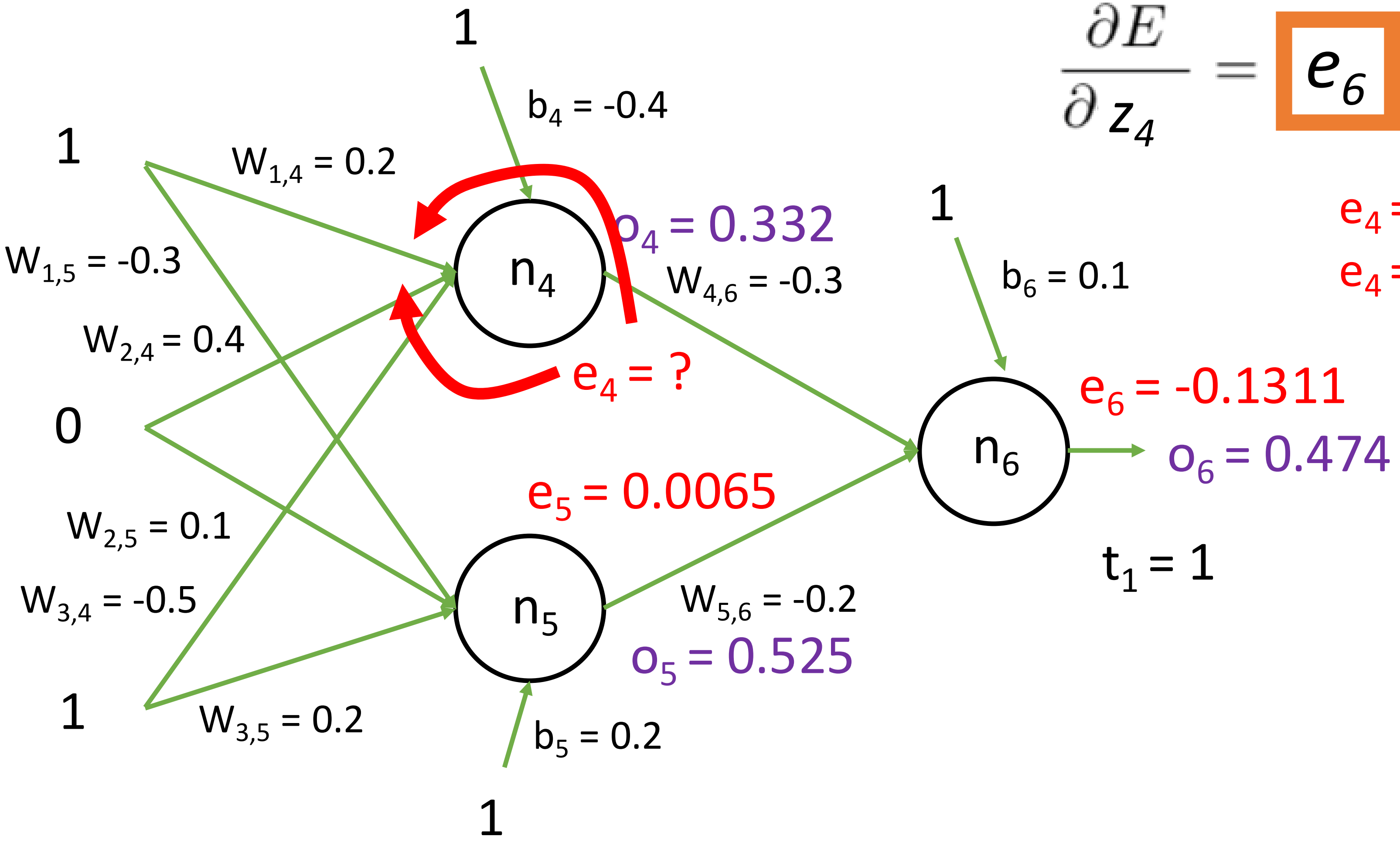
$$\frac{\partial E}{\partial z_4} = \boxed{e_6}\,\boxed{w_{4,6}}\,\boxed{(1 - o_4\;))\; o_4}$$

For node 5, we will first compute:

$$\boxed{\frac{\partial E}{\partial z_4}} = \boxed{\frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6}} * \boxed{\frac{\partial z_6}{\partial o_4}} * \boxed{\frac{\partial o_4}{\partial z_4}}$$

Image source: https://www.v7labs.com/blog/neural-networks-activation-functions

# Example: Step 3 – Backward Pass

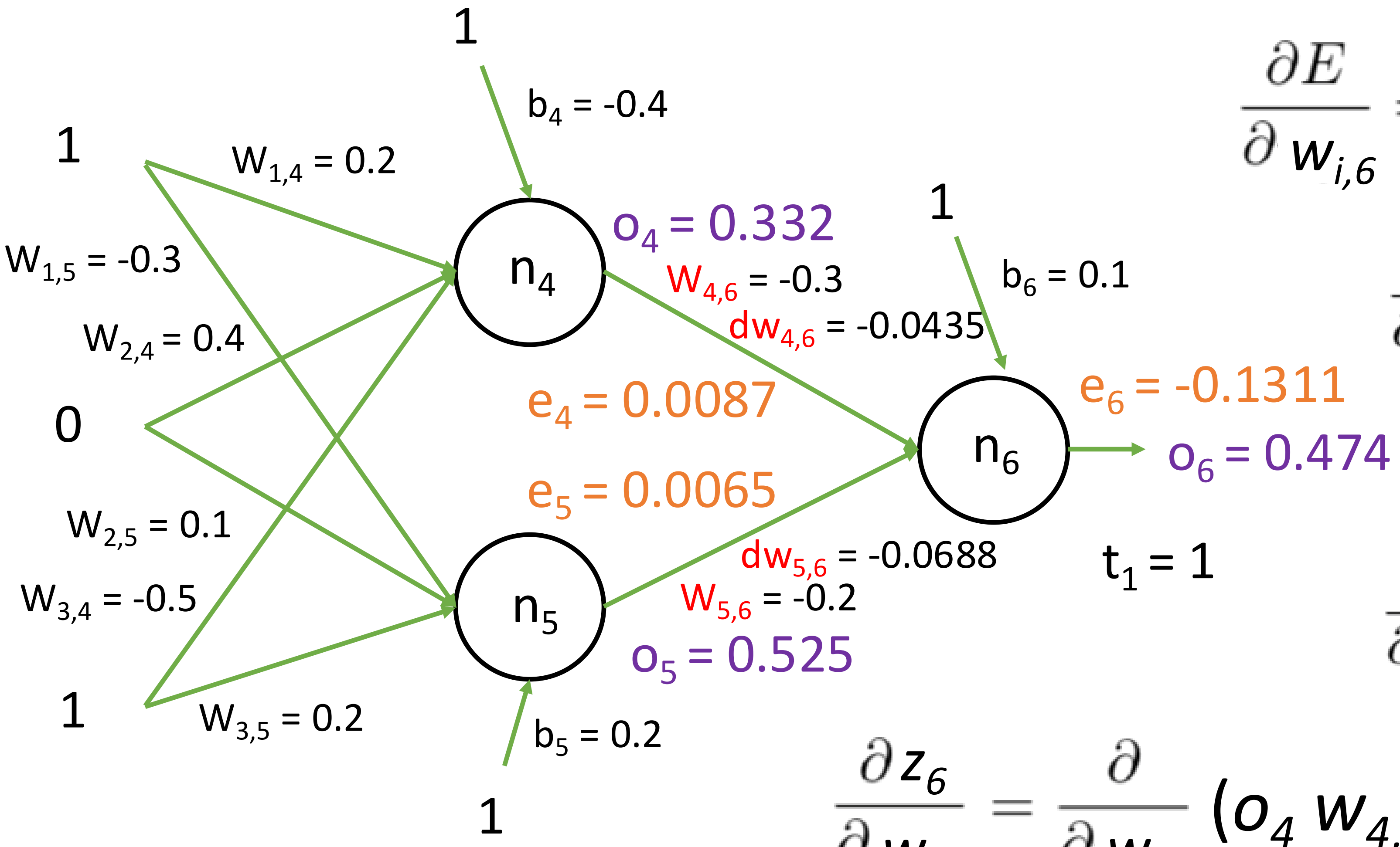



$$\frac{\partial E}{\partial z_4} = \boxed{e_6}\,\boxed{w_{4,6}}\,\boxed{(1 - o_4)\,o_4}$$

$e_4 = -0.1311 * -0.3 * (1-0.332) * 0.332$

$e_4 = 0.0087$

# Example: Step 3 – Backward Pass




$$\frac{\partial E}{\partial w_{i,6}} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6} * \boxed{\frac{\partial z_6}{\partial w_{i,6}}}$$

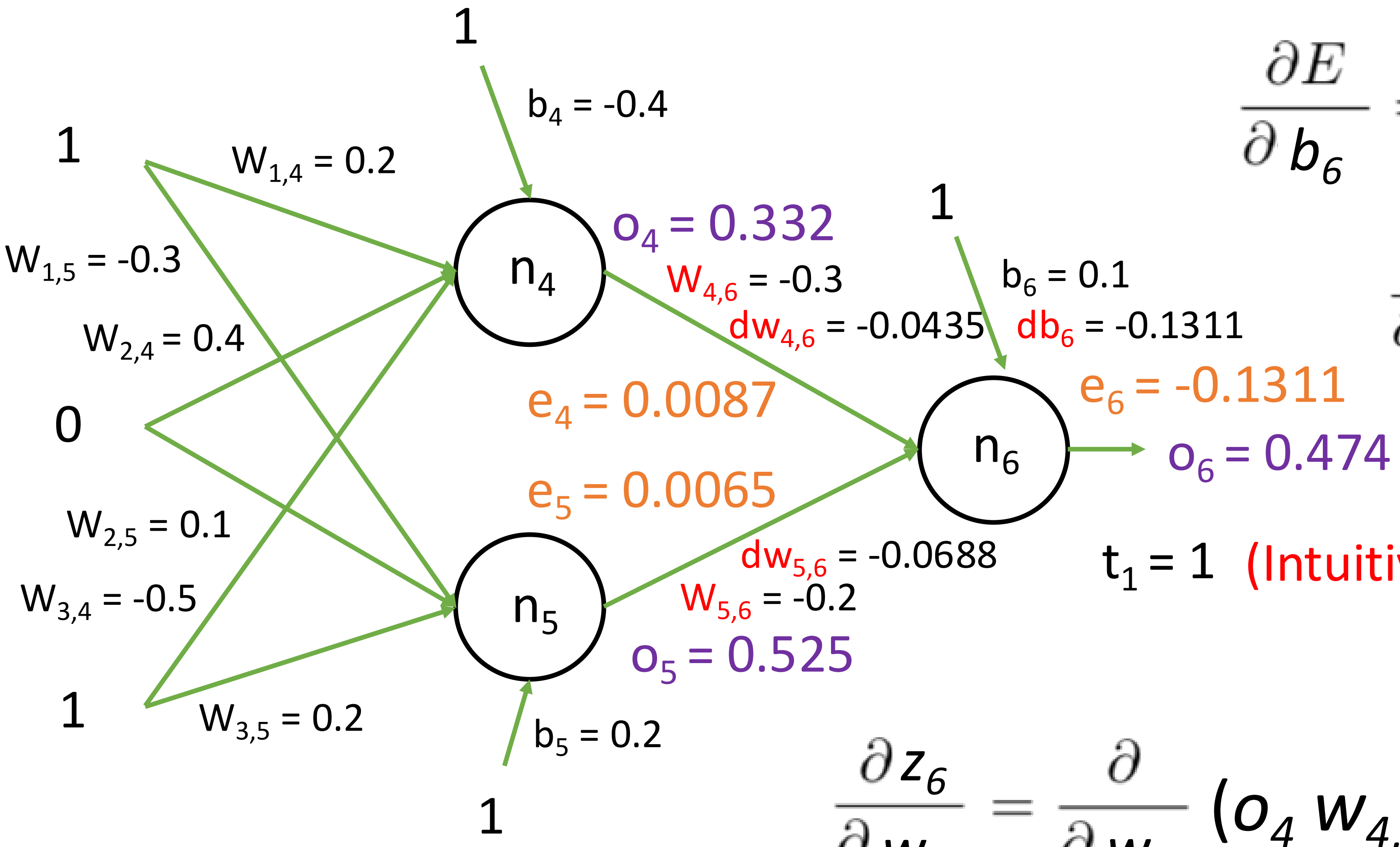
$$\frac{\partial E}{\partial w_{4,6}} = e_6 * o_4 = -0.0435$$

*Recall, gradient points to direction of larger value!*

$$\frac{\partial E}{\partial w_{5,6}} = e_6 * o_5 = -0.0688$$

$$\frac{\partial z_6}{\partial w_{i,6}} = \frac{\partial}{\partial w_{i,6}}(o_4 w_{4,6} + o_5 w_{5,6} + b_6) = \boxed{o_i}$$

# Example: Step 3 – Backward Pass



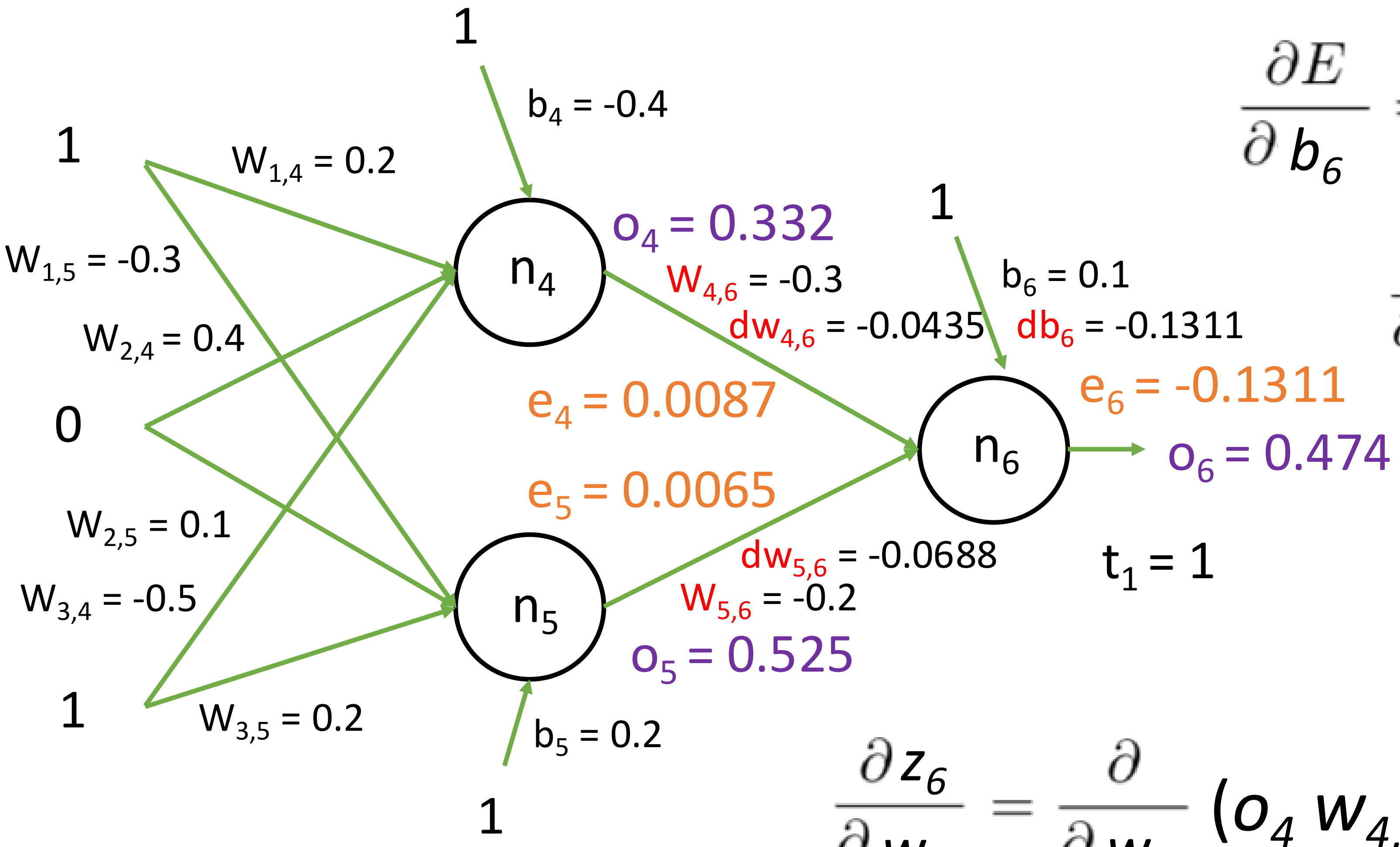

$$\frac{\partial E}{\partial b_6} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6} * \frac{\partial z_6}{\partial b_6}$$

$$\frac{\partial E}{\partial b_6} = e_6 * 1 = -0.1311$$

(Intuitively, just passes error along)

$$\frac{\partial z_6}{\partial w_{i,6}} = \frac{\partial}{\partial w_{i,6}}(o_4 w_{4,6} + o_5 w_{5,6} + b_6) = 1$$

# Example: Step 3 – Backward Pass



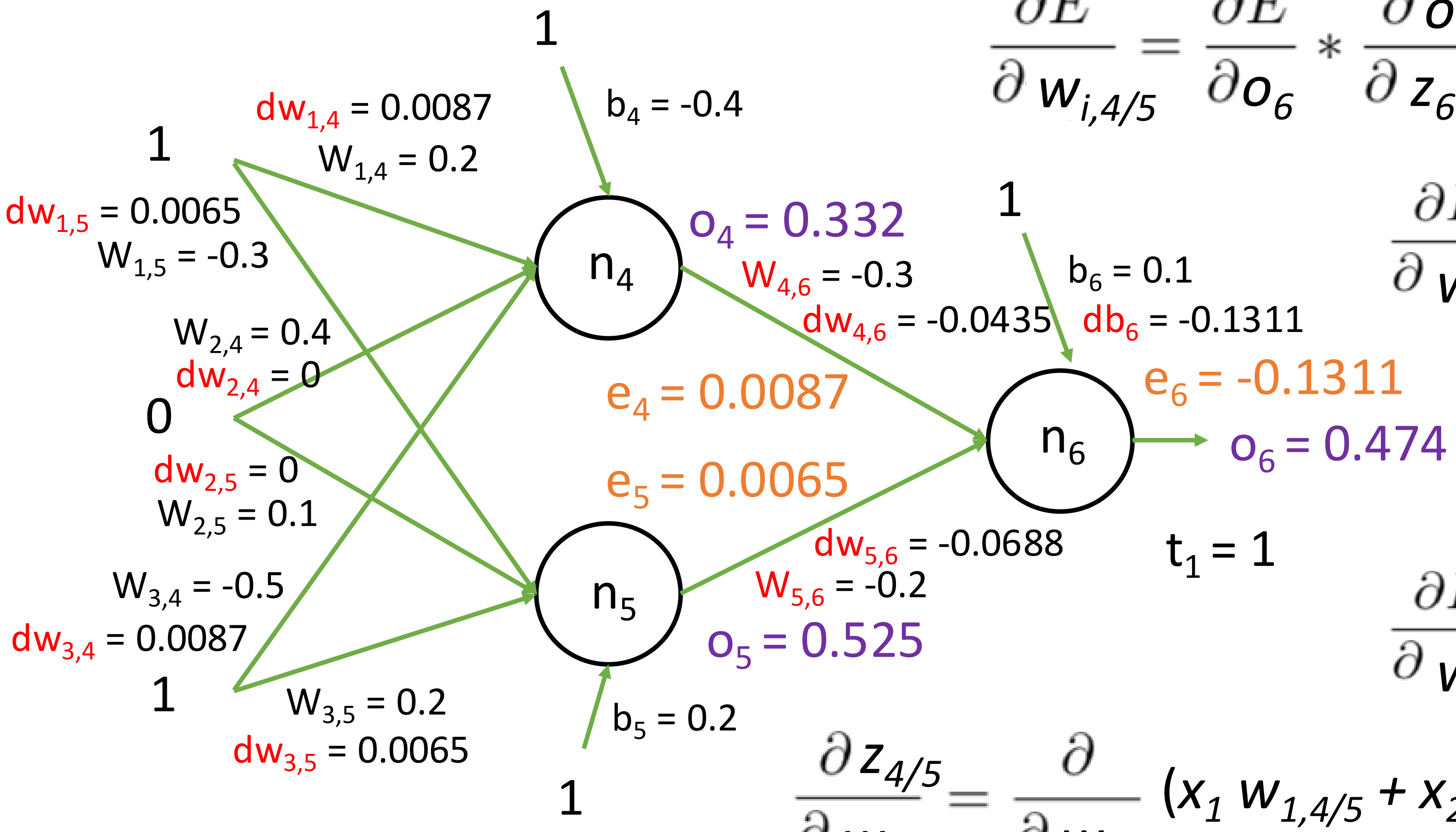

$$\frac{\partial E}{\partial b_6} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6} * \frac{\partial z_6}{\partial b_6}$$

$$\frac{\partial E}{\partial b_6} = e_6 * 1 = -0.1311$$

*Recall, gradient points to direction of larger value!*

$$\frac{\partial z_6}{\partial w_{i,6}} = \frac{\partial}{\partial w_{i,6}}(o_4 w_{4,6} + o_5 w_{5,6} + b_6) = 1$$

# Example: Step 3 – Backward Pass




$$\frac{\partial E}{\partial w_{i,4/5}} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6} * \frac{\partial z_6}{\partial o_{4/5}} * \frac{\partial o_{4/5}}{\partial z_{4/5}} * \frac{\partial z_{4/5}}{\partial w_{i,4/5}}$$

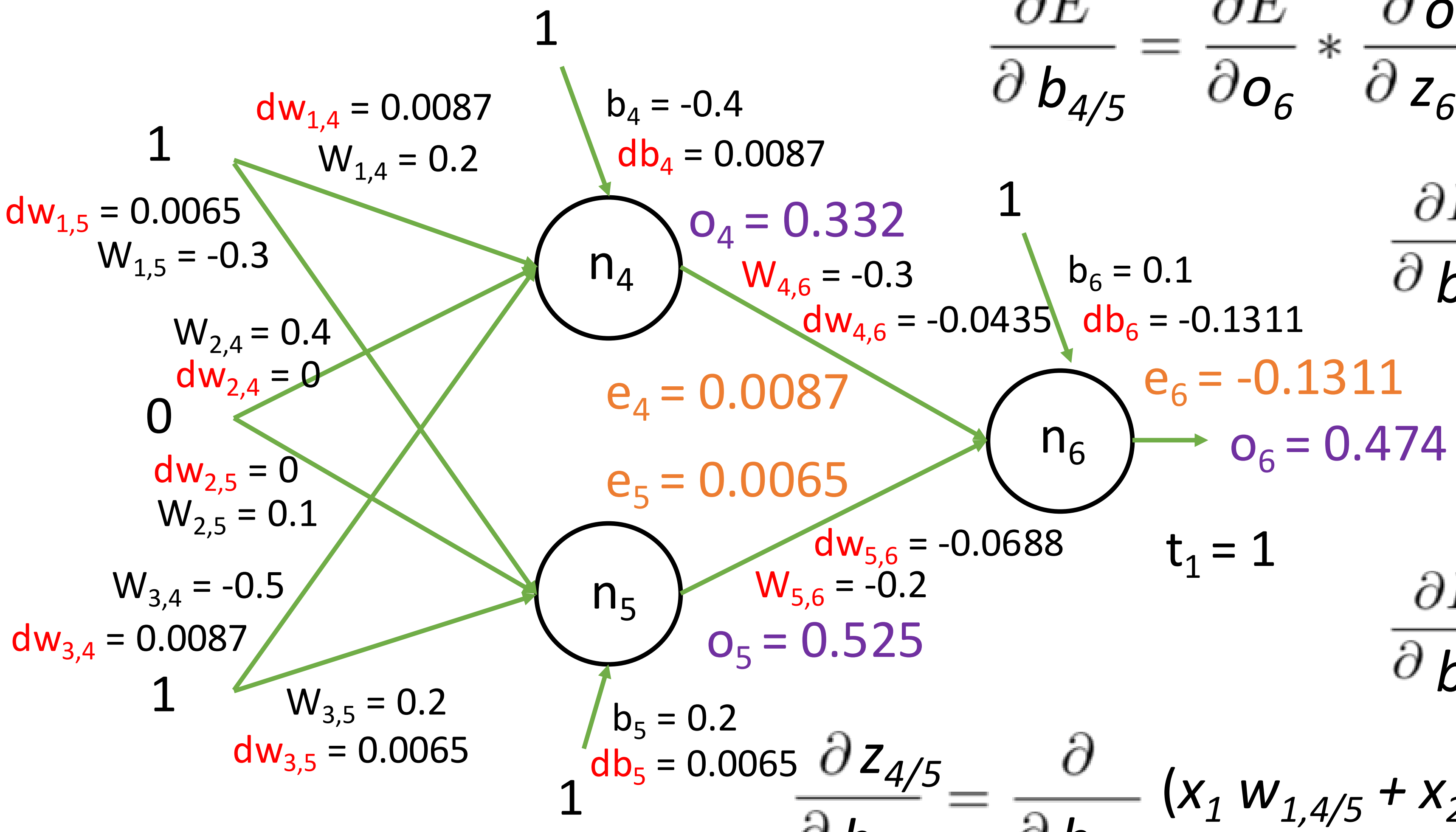
$$\frac{\partial E}{\partial w_{1,4}} = e_4 * x_1 = 0.0087$$

*Recall, gradient points to direction of larger value!*

$$\frac{\partial E}{\partial w_{3,5}} = e_5 * x_3 = 0.0065$$

$$\frac{\partial z_{4/5}}{\partial w_{i,4/5}} = \frac{\partial}{\partial w_{i,4/5}} (x_1 w_{1,4/5} + x_2 w_{2,4/5} + x_3 w_{3,4/5} + b_{4/5}) = x_i$$

# Example: Step 3 – Backward Pass




$$\frac{\partial E}{\partial b_{4/5}} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6} * \frac{\partial z_6}{\partial o_{4/5}} * \frac{\partial o_{4/5}}{\partial z_{4/5}} * \frac{\partial z_{4/5}}{\partial b_{4/5}}$$

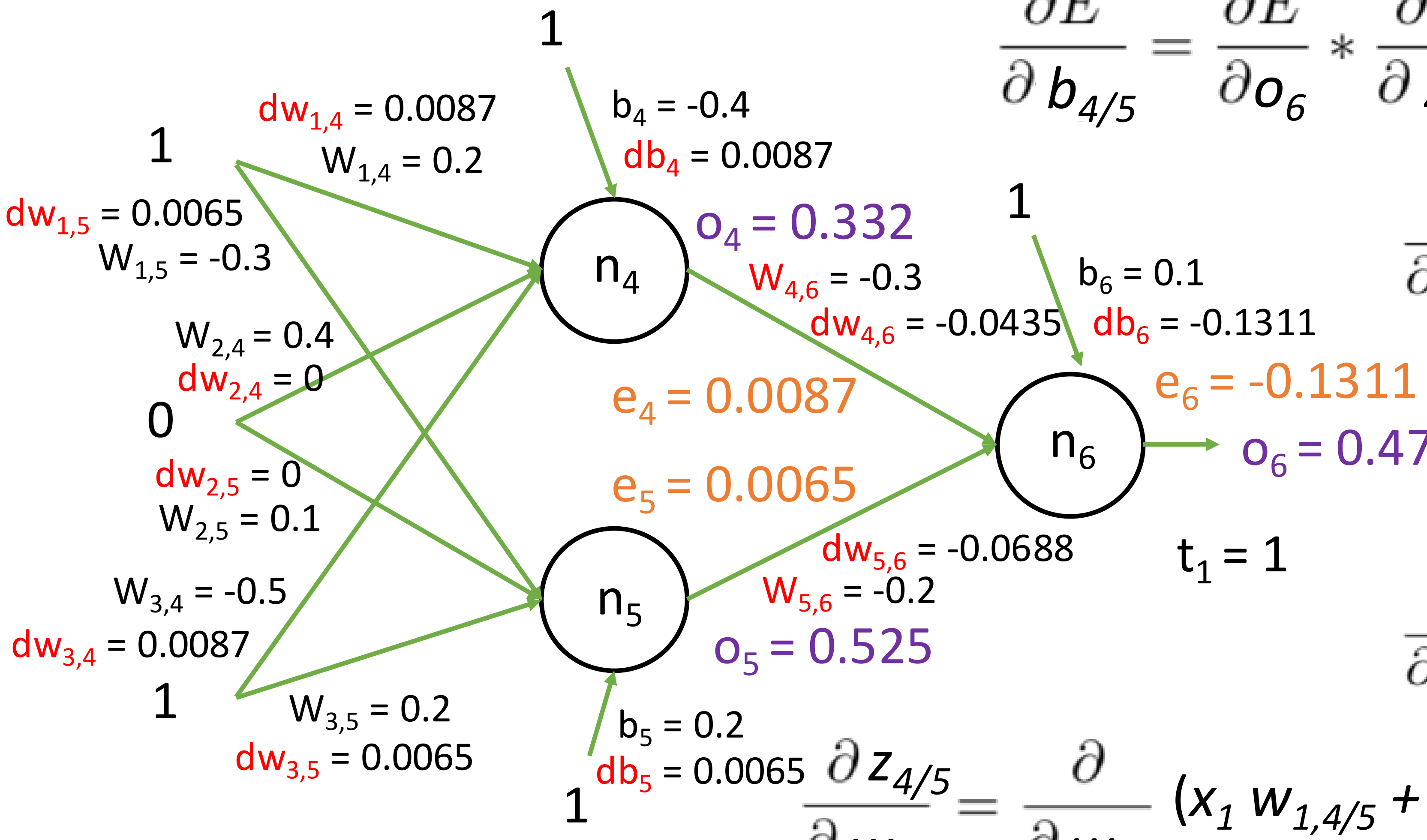
$$\frac{\partial E}{\partial b_4} = e_4 * 1 = 0.0087$$

(Just passes error along)

$$\frac{\partial E}{\partial b_5} = e_5 * 1 = 0.0065$$

$$\frac{\partial z_{4/5}}{\partial b_{4/5}} = \frac{\partial}{\partial b_{4/5}} (x_1 w_{1,4/5} + x_2 w_{2,4/5} + x_3 w_{3,4/5} + b_{4/5}) = 1$$

# Example: Step 3 – Backward Pass




$$\frac{\partial E}{\partial b_{4/5}} = \frac{\partial E}{\partial o_6} * \frac{\partial o_6}{\partial z_6} * \frac{\partial z_6}{\partial o_{4/5}} * \frac{\partial o_{4/5}}{\partial z_{4/5}} * \frac{\partial z_{4/5}}{\partial b_{4/5}}$$

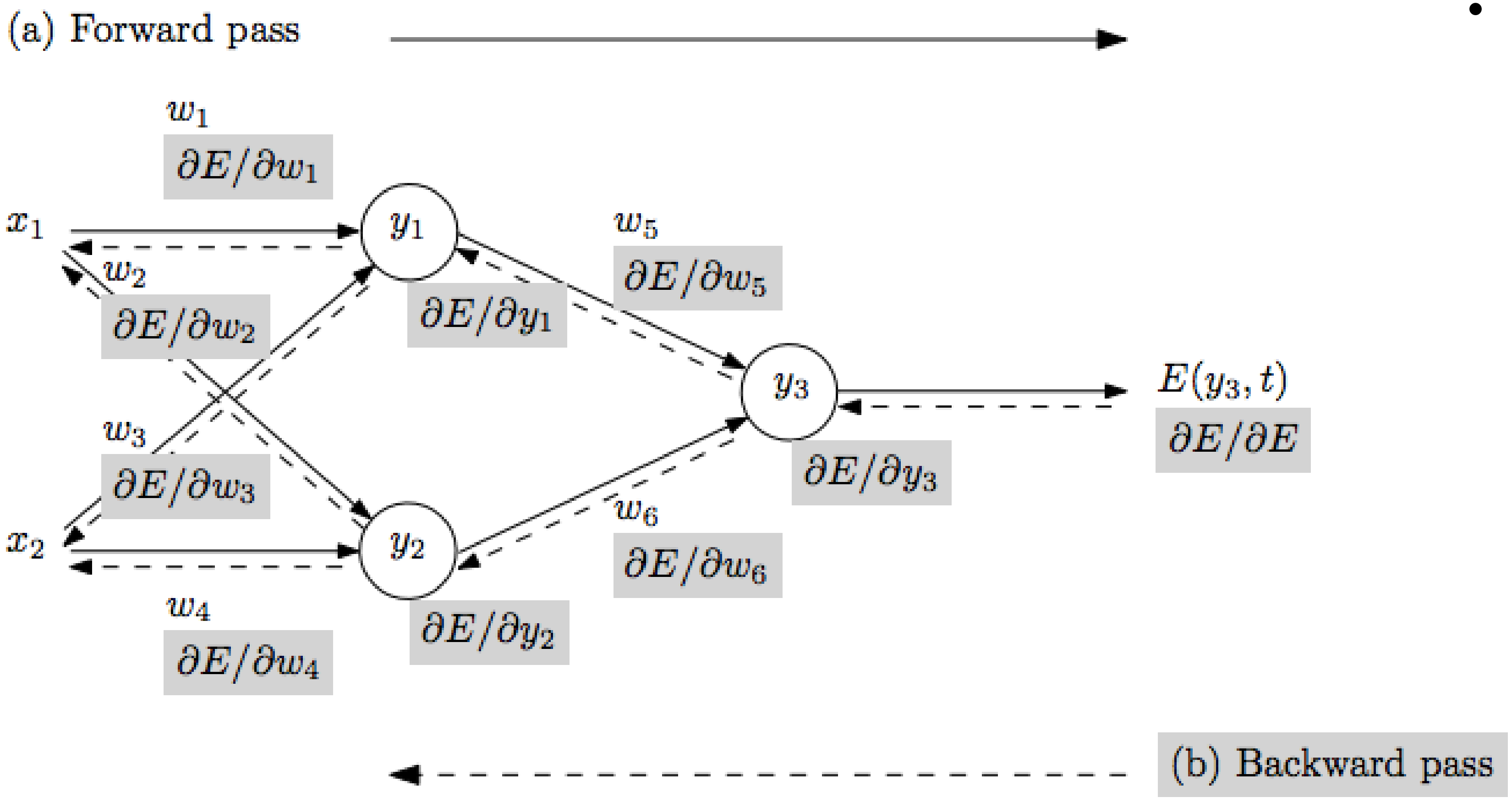
$$\frac{\partial E}{\partial b_4} = e_4 * 1 = 0.0087$$

*Recall, gradient points to direction of larger value!*

$$\frac{\partial E}{\partial b_5} = e_5 * 1 = 0.0065$$

$$\frac{\partial z_{4/5}}{\partial w_{i,4/5}} = \frac{\partial}{\partial w_{i,4/5}}(x_1 w_{1,4/5} + x_2 w_{2,4/5} + x_3 w_{3,4/5} + b_{4/5}) = 1$$

# Example: Step 4 – Update Weights




- Repeat until stopping criterion met:
  1. **Forward pass**: propagate training data through model to make predictions
  2. **Error quantification**: measure error of the model's predictions on training data using a loss function
  3. **Backward pass**: calculate gradients to determine how each model parameter contributed to model error
  4. Update each parameter using calculated gradients

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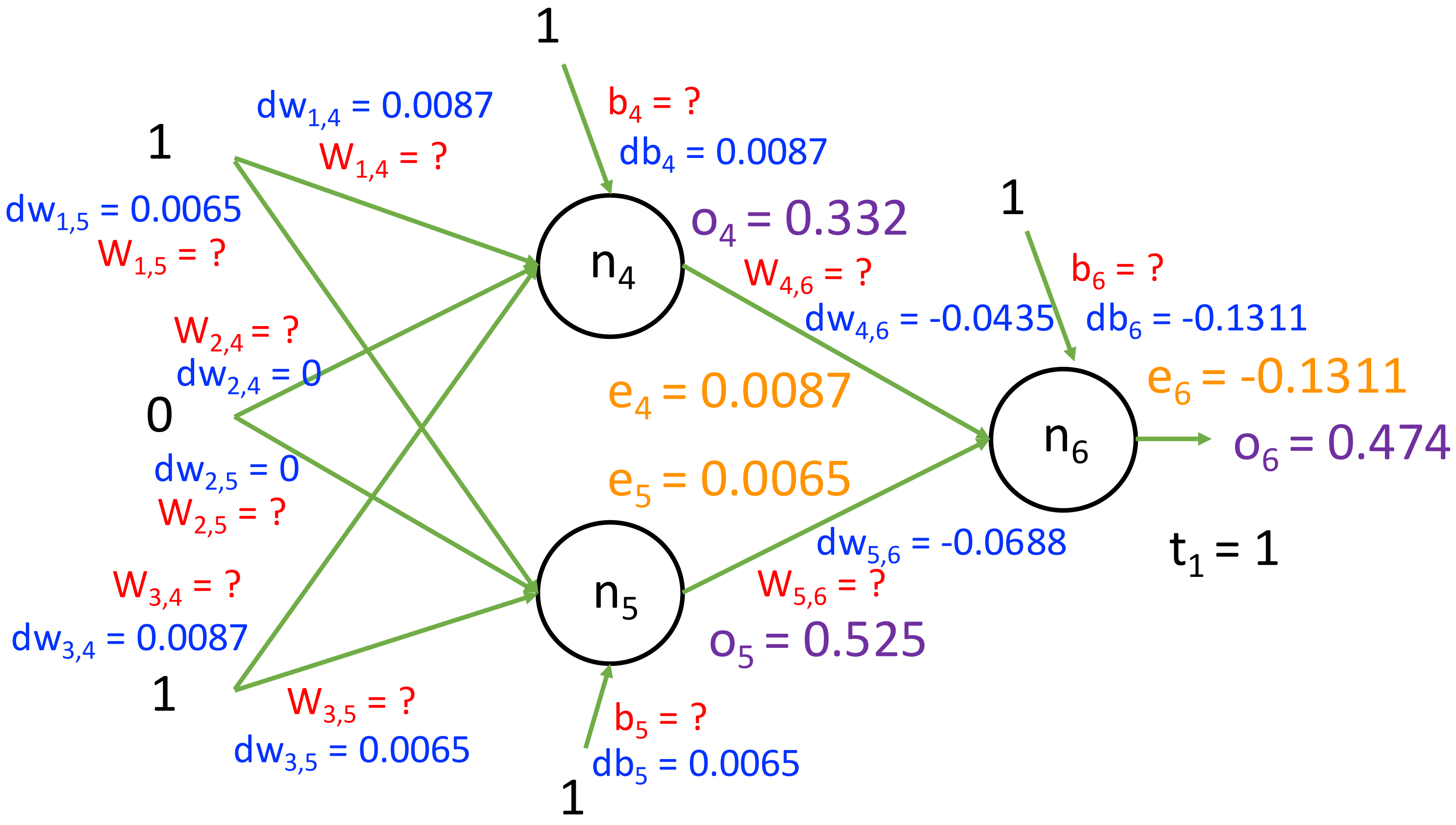
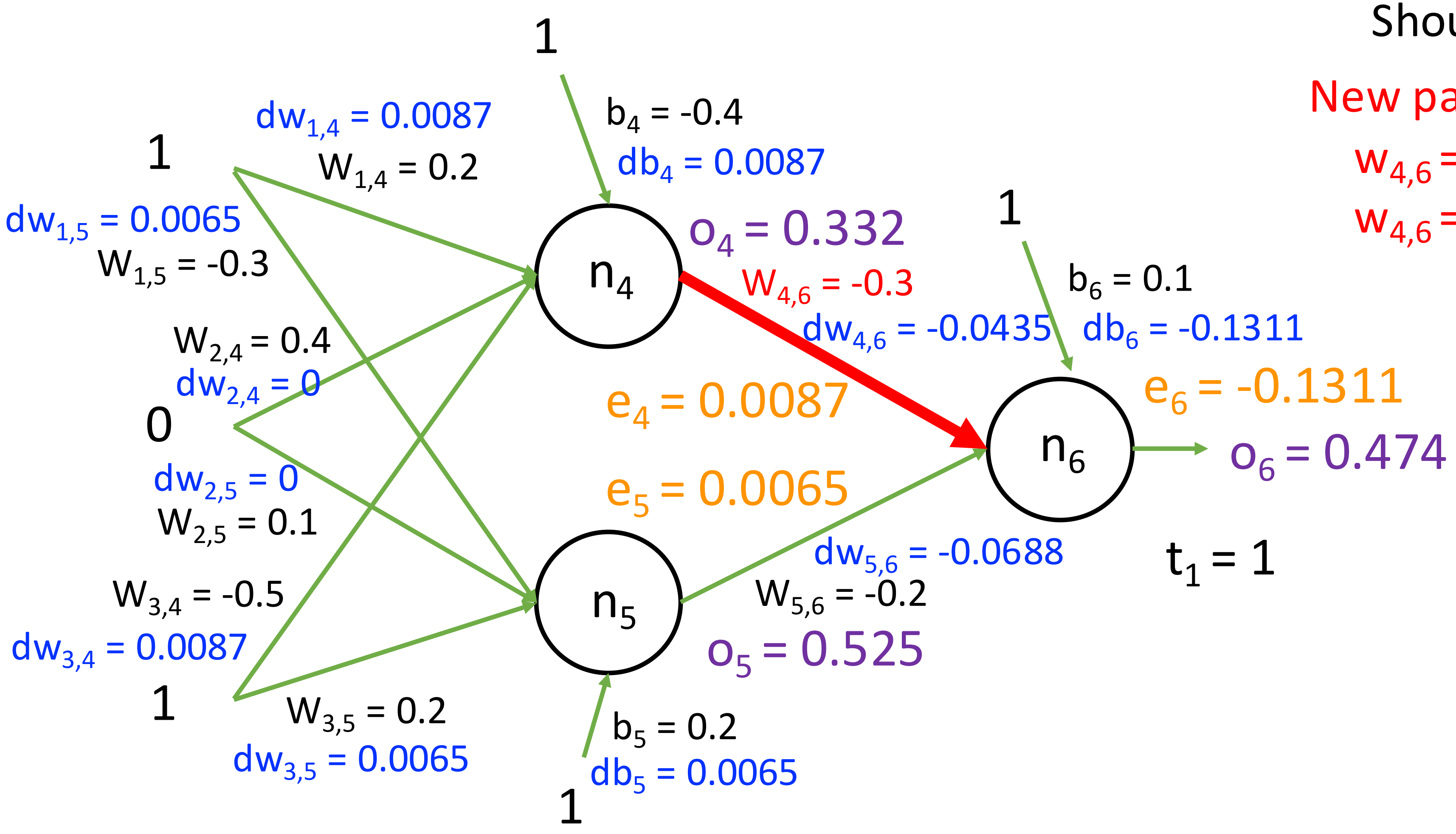
# Example: Step 4 – Update Weights



# Example: Step 4 – Update Weights



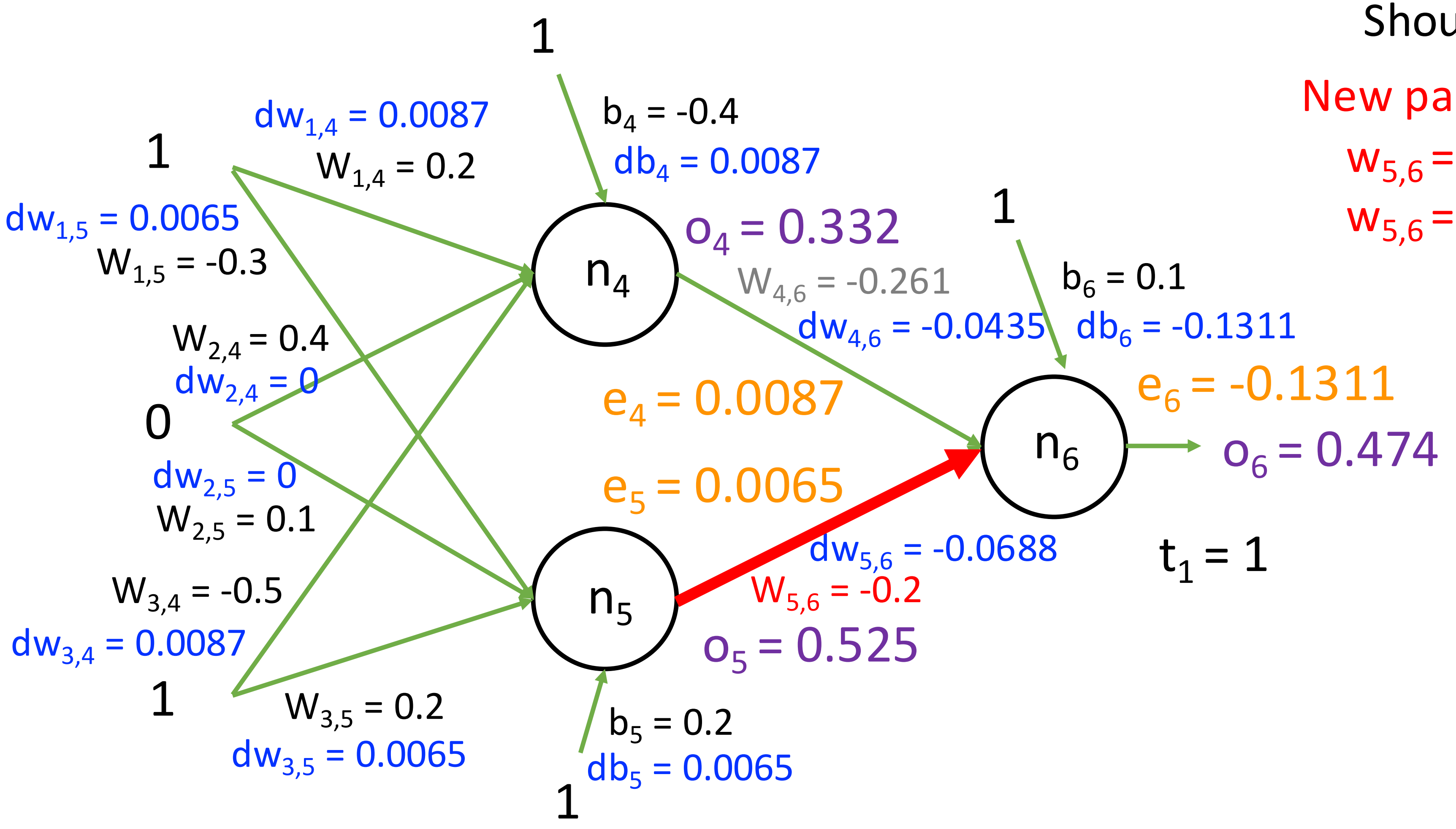

Should $w_{4,6}$ increase or decrease?

New parameters (learning rate = 0.9):

$w_{4,6} = -0.3 - 0.9 * -0.0435$

$w_{4,6} = -0.261$

# Example: Step 4 – Update Weights



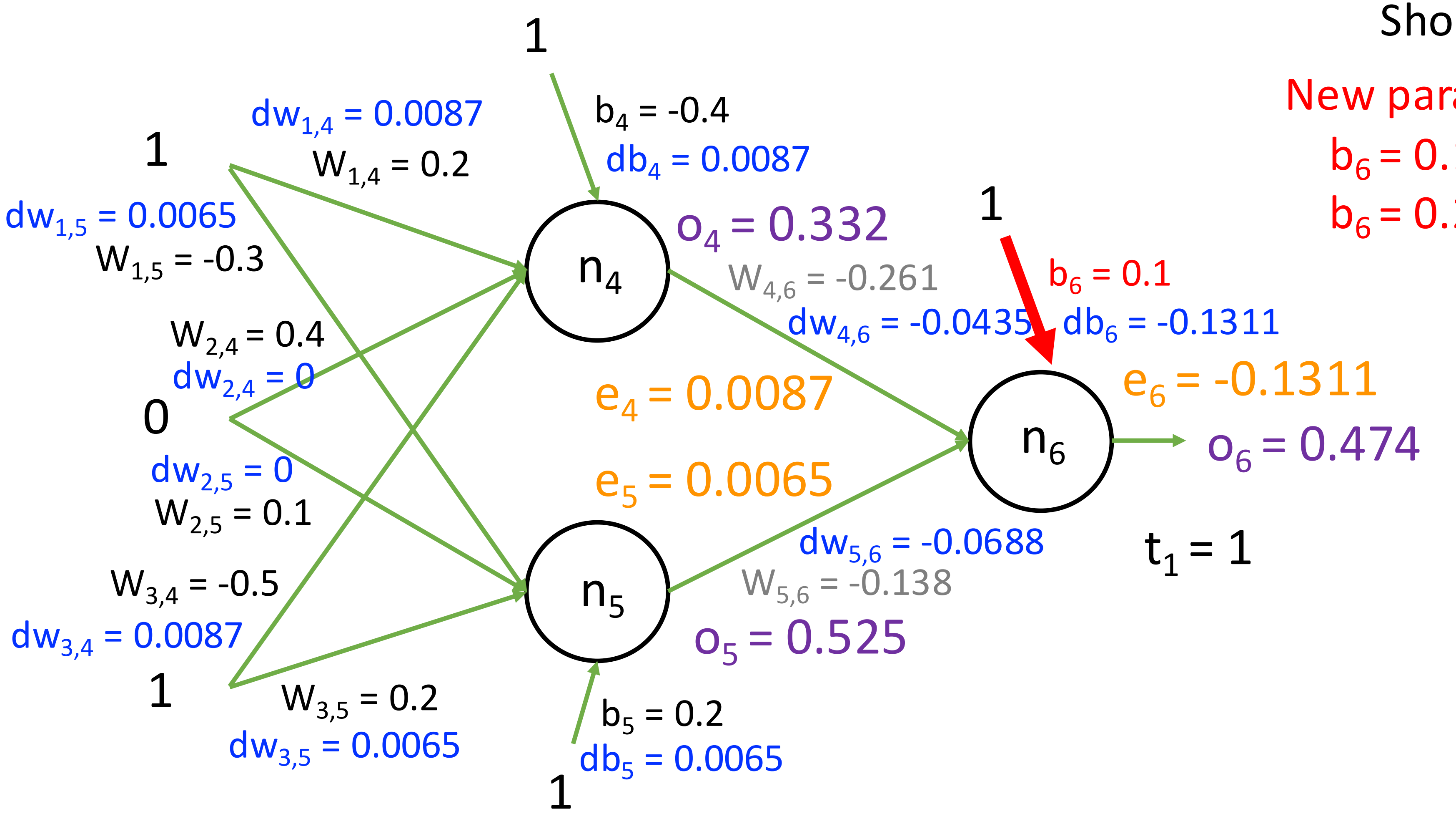

Should $w_{5,6}$ increase or decrease?

New parameters (learning rate = 0.9):

$w_{5,6} = -0.2 - 0.9 * -0.0688$

$w_{5,6} = -0.138$

# Example: Step 4 – Update Weights



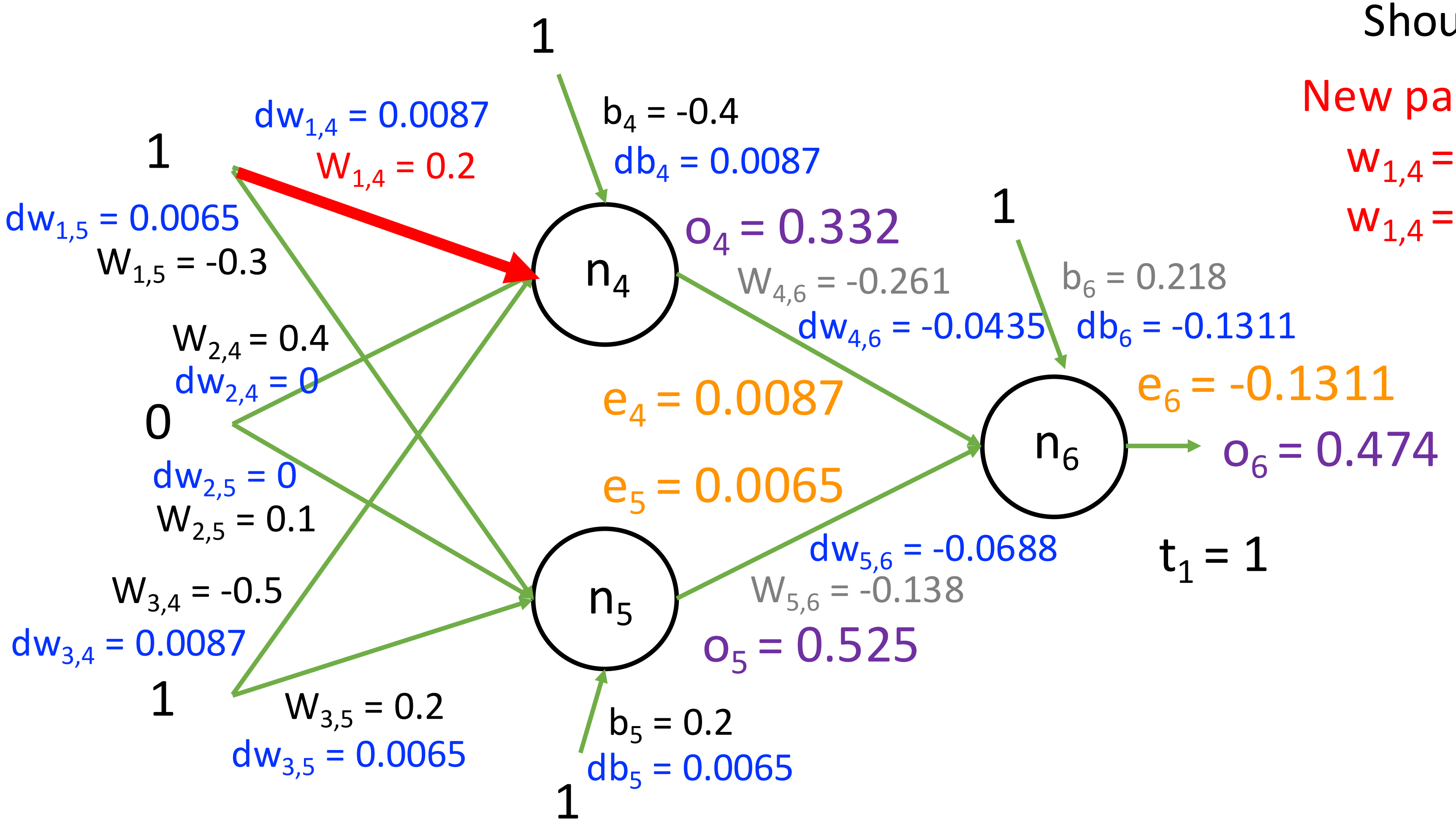

Should $b_6$ increase or decrease?

New parameters (learning rate = 0.9):

$b_6 = 0.1 - 0.9 * -0.1311$

$b_6 = 0.218$

# Example: Step 4 – Update Weights



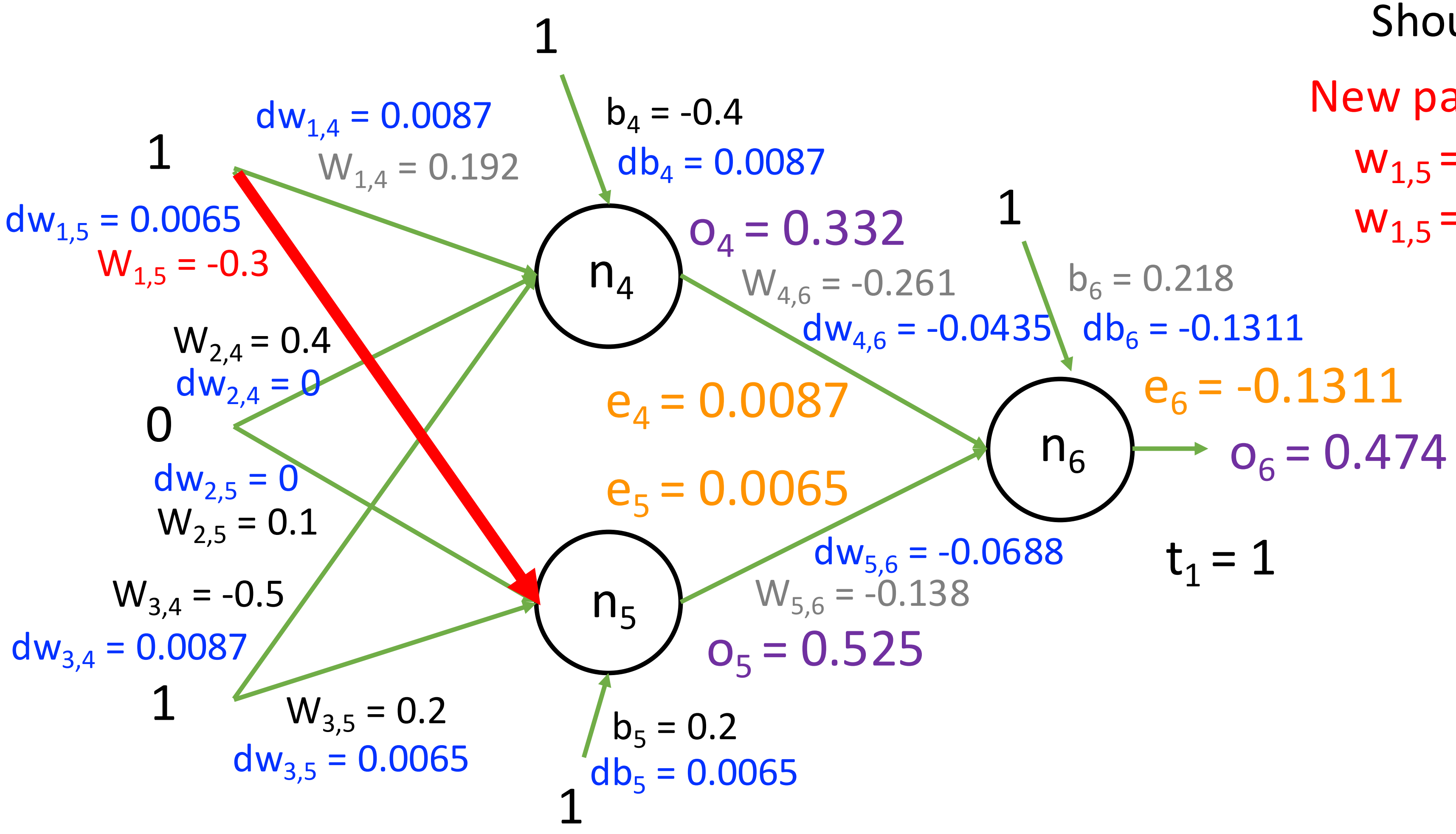

Should $w_{1,4}$ increase or decrease?

New parameters (learning rate = 0.9):

$w_{1,4} = 0.2 - 0.9 * 0.0087$

$w_{1,4} = 0.192$

# Example: Step 4 – Update Weights



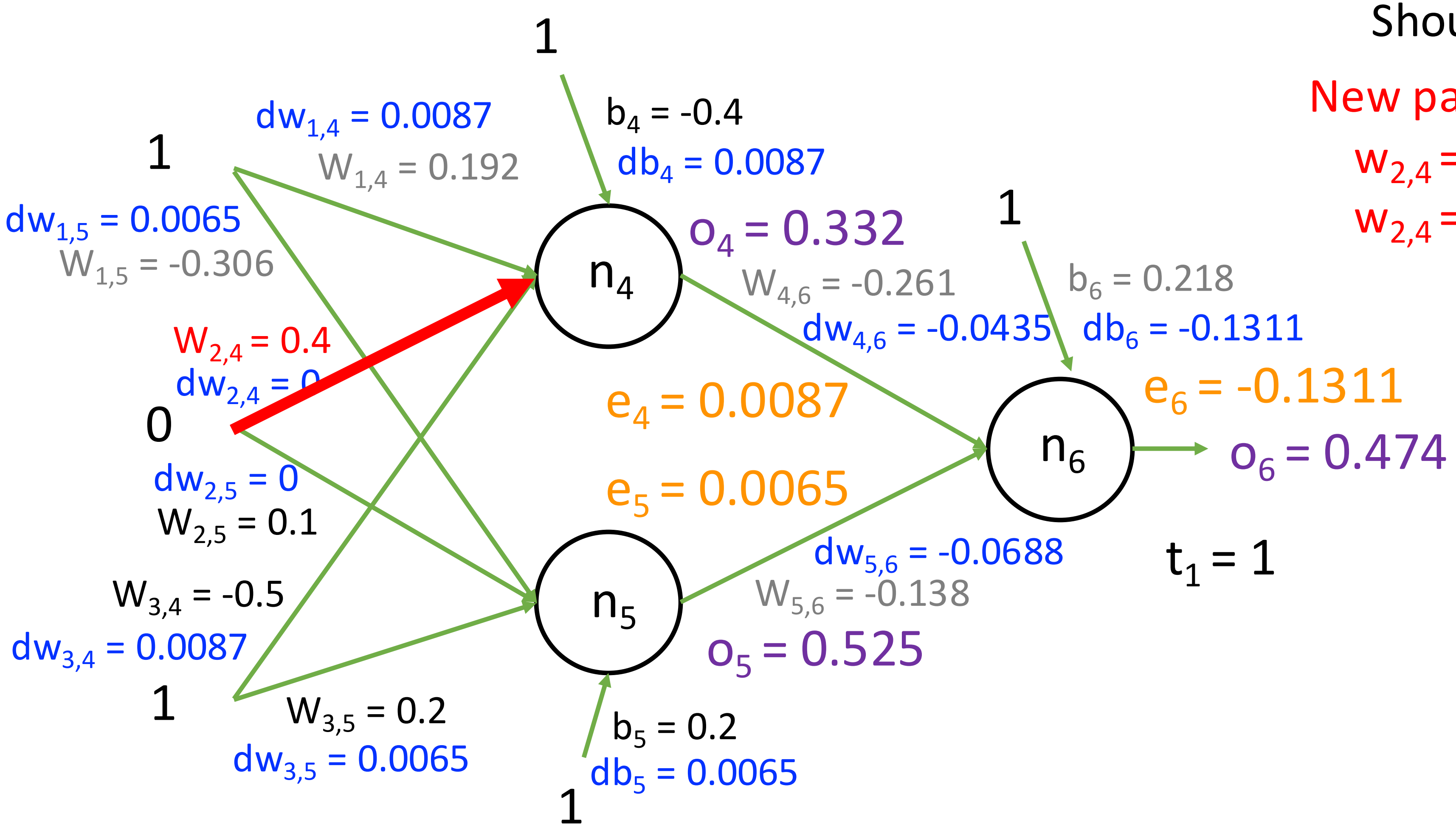

Should $w_{1,5}$ increase or decrease?

New parameters (learning rate = 0.9):

$w_{1,5}$ = -0.3 - 0.9 * 0.0065

$w_{1,5}$ = -0.306

# Example: Step 4 – Update Weights



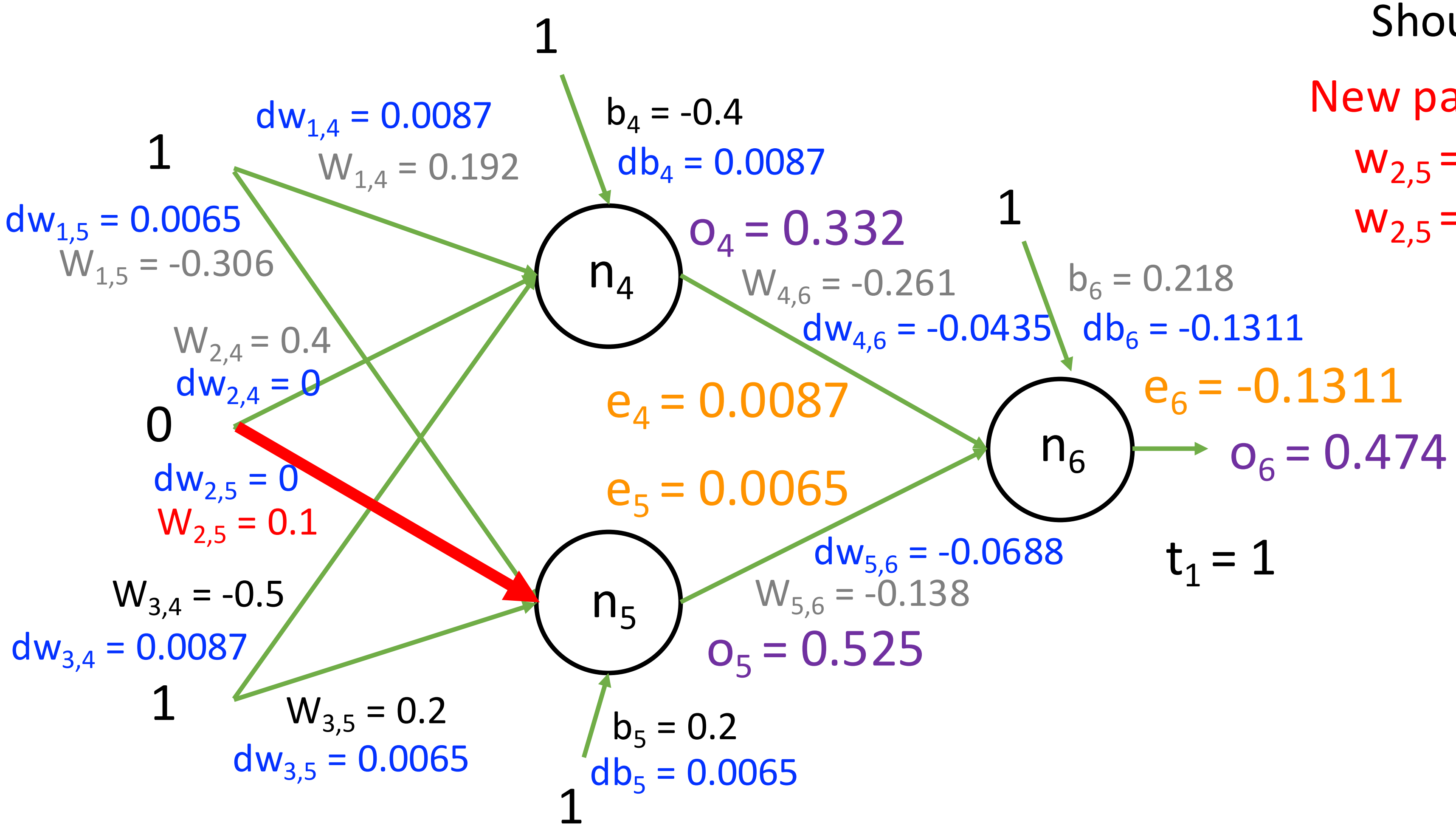

Should $w_{2,4}$ increase or decrease?

New parameters (learning rate = 0.9):

$w_{2,4} = 0.4 - 0.9 * 0$

$w_{2,4} = 0.4$

# Example: Step 4 – Update Weights



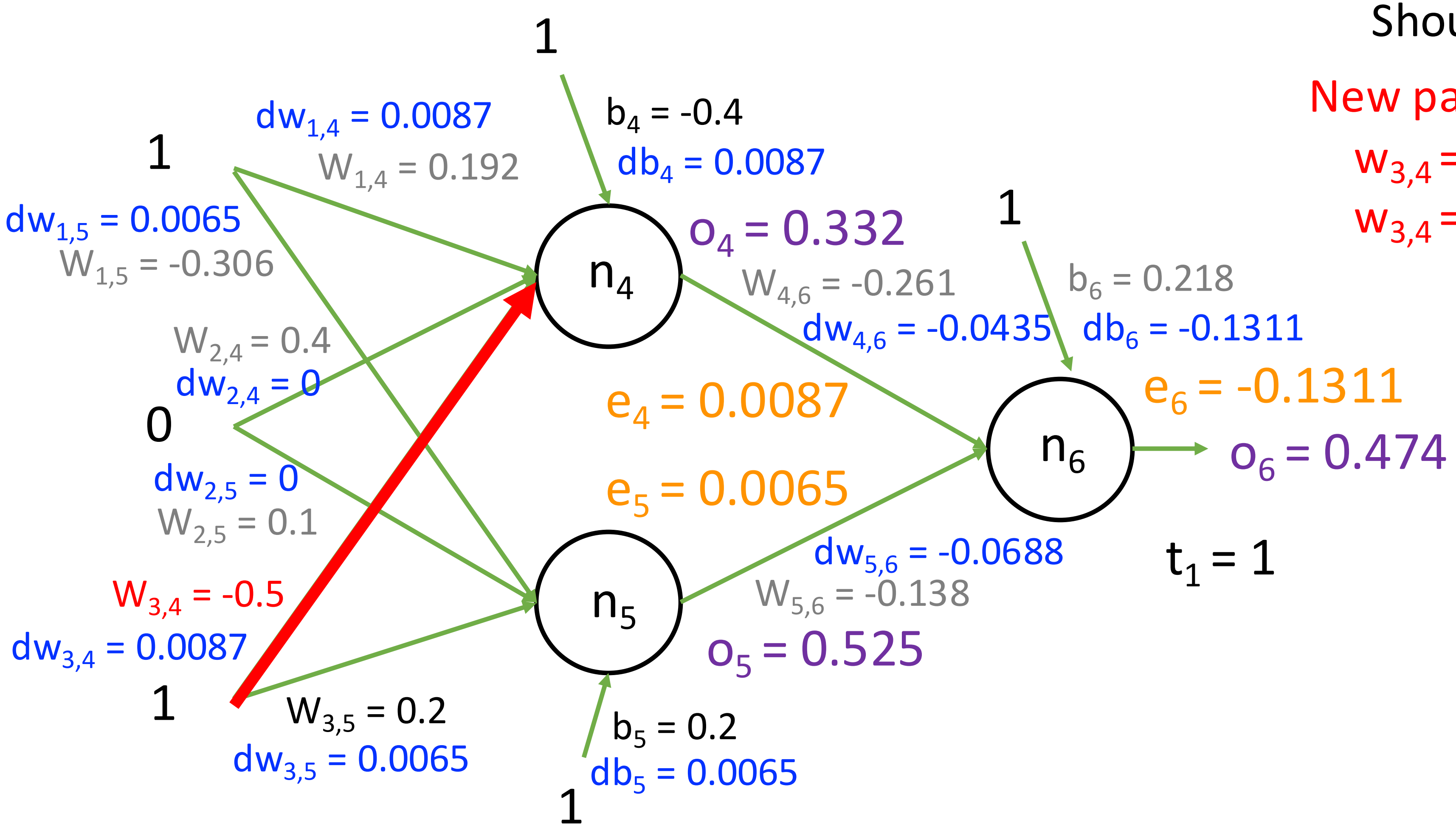

Should $w_{2,5}$ increase or decrease?

New parameters (learning rate = 0.9):

$w_{2,5} = 0.1 - 0.9 * 0$

$w_{2,5} = 0.1$

# Example: Step 4 – Update Weights



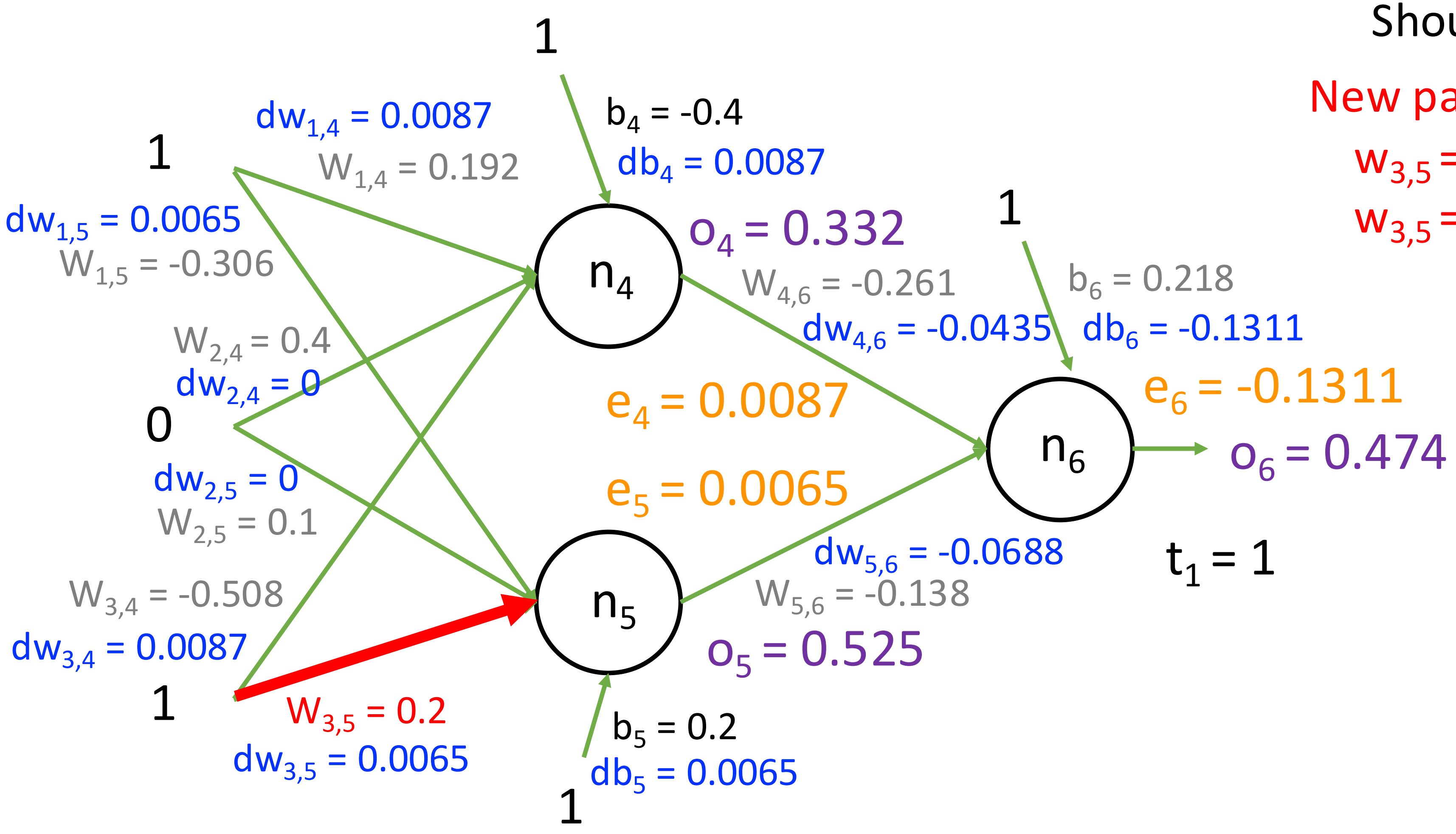

Should $w_{3,4}$ increase or decrease?

New parameters (learning rate = 0.9):

$w_{3,4}$ = -0.5 - 0.9 * 0.0087

$w_{3,4}$ = -0.508

# Example: Step 4 – Update Weights



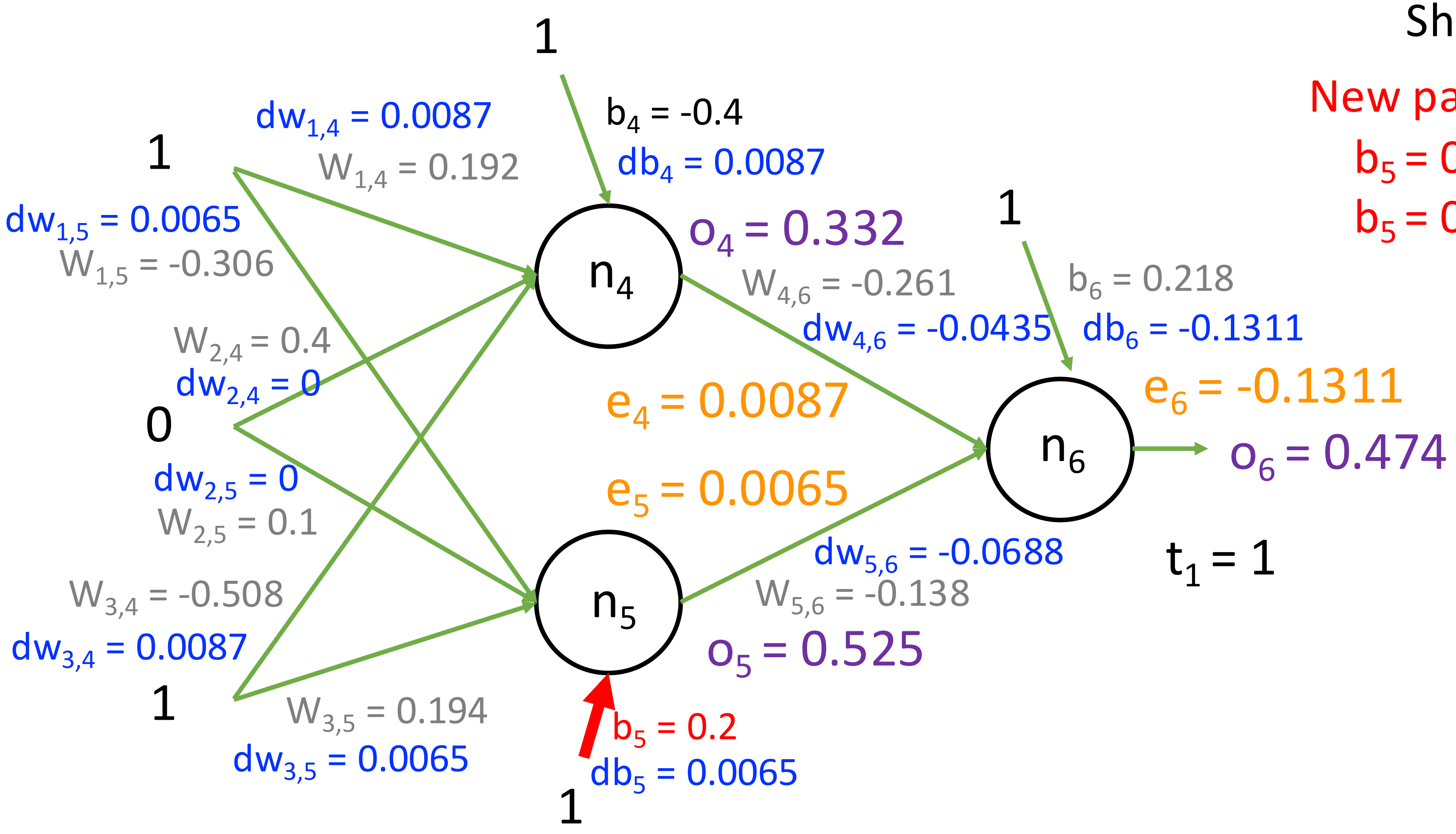

Should $w_{3,5}$ increase or decrease?

New parameters (learning rate = 0.9):

$w_{3,5} = 0.2 - 0.9 * 0.0065$

$w_{3,5} = 0.194$

# Example: Step 4 – Update Weights



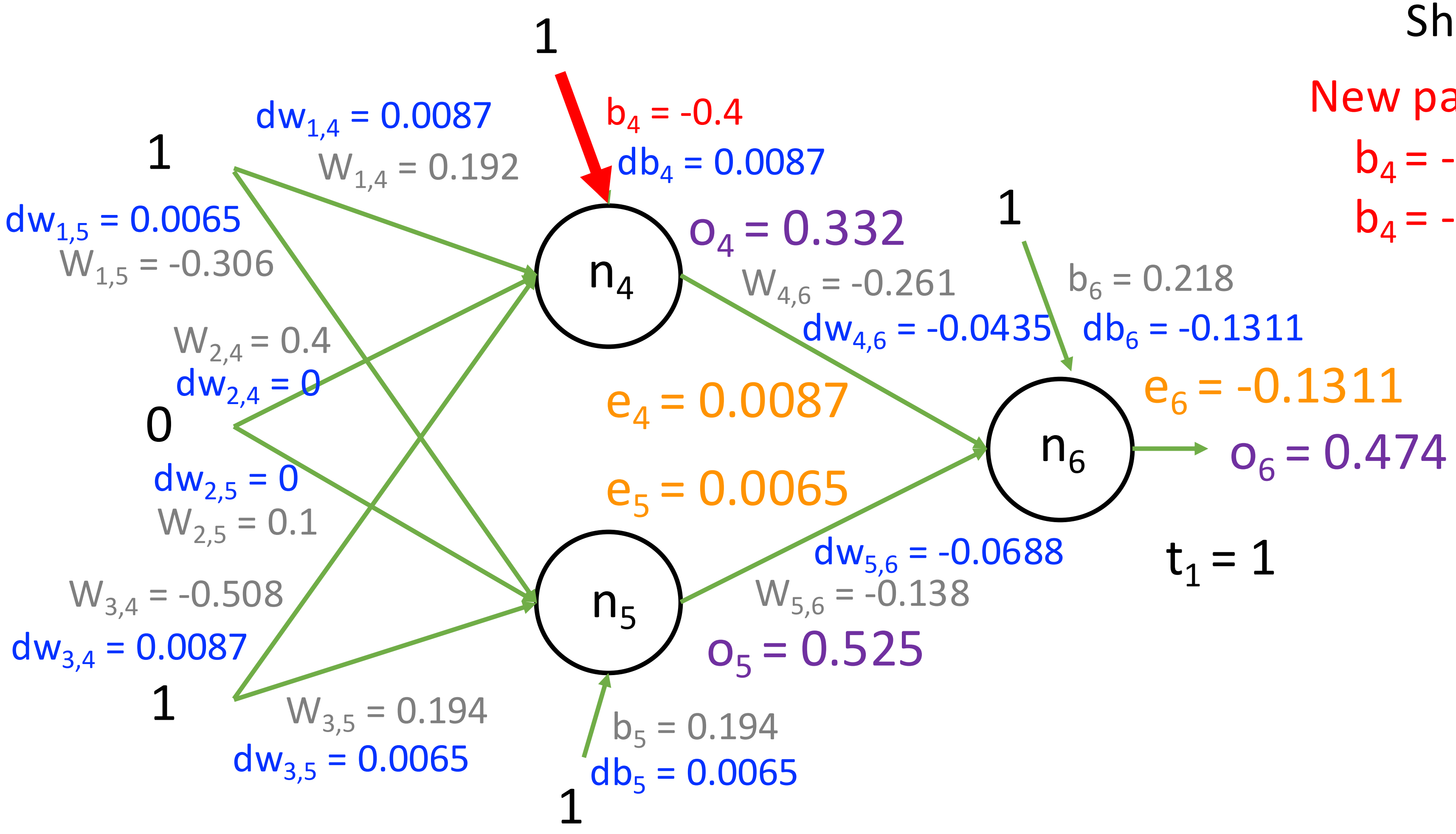

Should $b_5$ increase or decrease?

New parameters (learning rate = 0.9):

$b_5 = 0.2 - 0.9 * 0.0065$

$b_5 = 0.194$

# Example: Step 4 – Update Weights



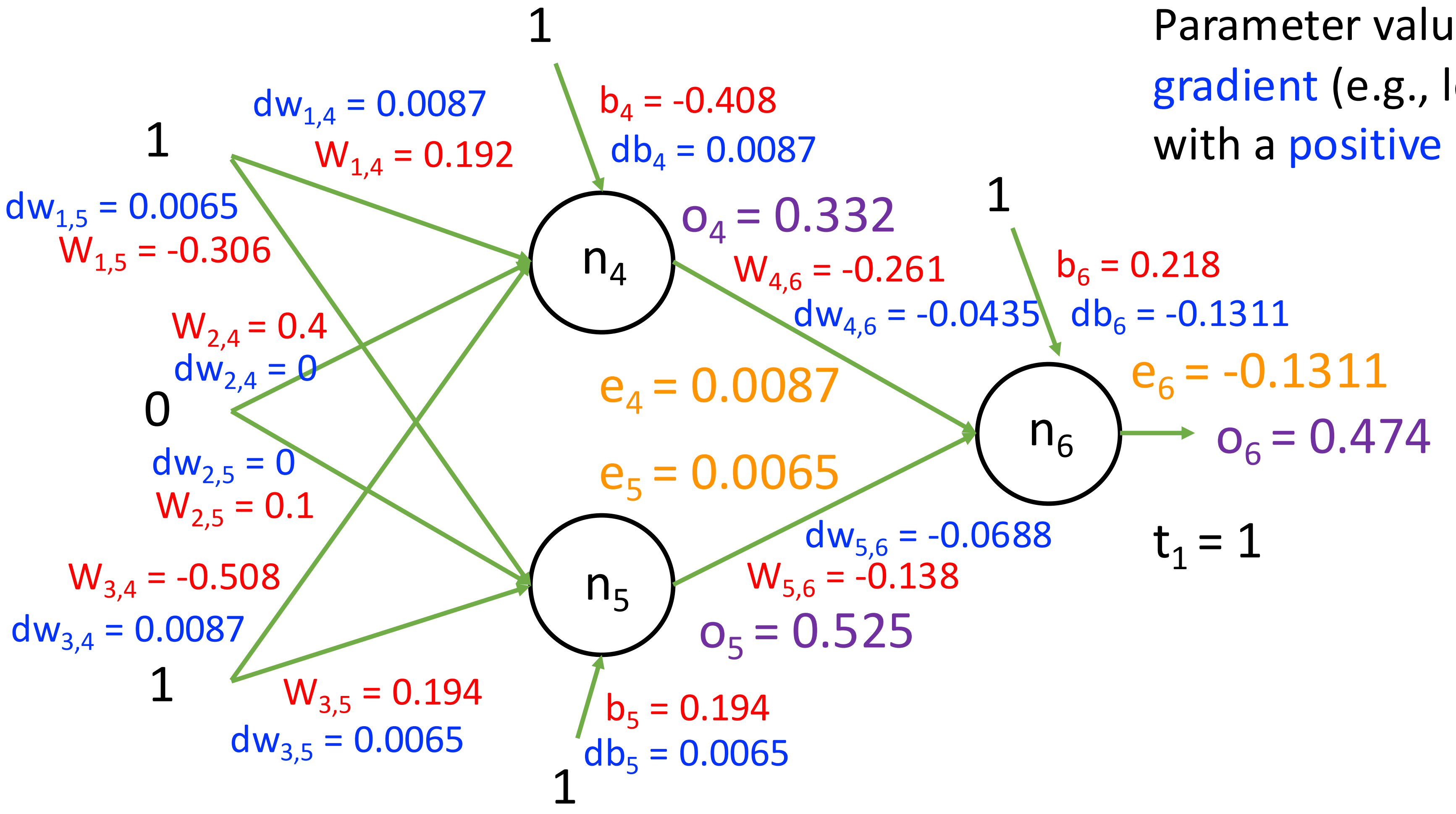

Should $b_4$ increase or decrease?

New parameters (learning rate = 0.9):

$b_4 = -0.4 - 0.9 * 0.0087$

$b_4 = -0.408$

# Example: Repeat Steps 1-4 With New Samples



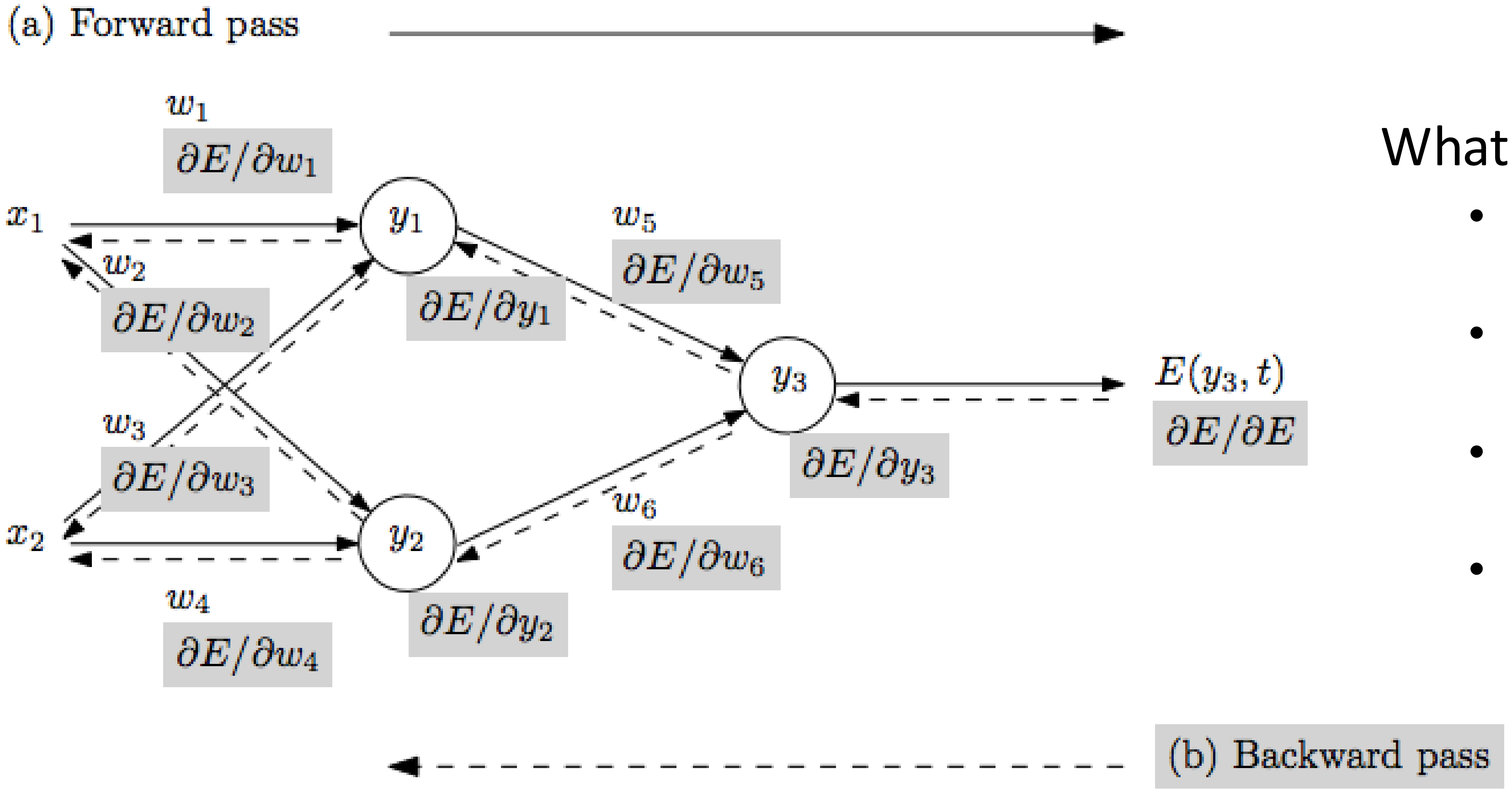

Parameter values **increase** with a negative gradient (e.g., less negative) and **decrease** with a positive gradient (e.g., more negative)

# Example: Completed Training Example For…

- Binary classification: predict if a student will get a B or better (minimum required for CS graduate student in this course)

- Inputs:
    - Fraction of assignments completed
    - Fraction of readings read
    - Fraction of lectures watched

# When to Stop Training Neural Networks?




What stopping criterion to use?

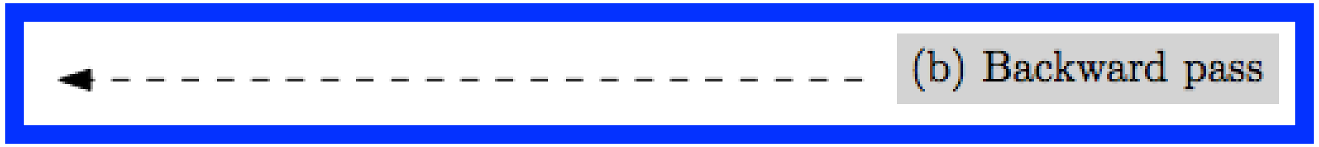
- Weight changes are incredibly small
- Finished a pre-specified number of epochs
- Percentage of misclassified example is below some threshold
- …

# Training Summary: How Neural Networks Learn

Calculating gradients depends on:

1) Objective/loss function

2) Activation functions




- Repeat until stopping criterion met:
  1. **Forward pass**: propagate training data through model to make predictions
  2. **Error quantification**: measure error of the model's predictions on training data using a loss function
  3. **Backward pass**: calculate gradients to determine how each model parameter contributed to model error
  4. Update each parameter using calculated gradients

Baydin et al. Automatic Differentiation in Machine Learning: a Survey. 2018

# Today’s Topics

- Gradient descent: how neural networks learn
- Mathematical foundation of gradient descent: derivatives
- Applying gradient descent to train neural networks
- Training example

The End