Controlled Supermartingale Functions for Stochastic Differential Equations: Inference and Applications.

Masoumeh Ghanbarpour, Guillaume O. Berger, and Sriram Sankaranarayanan

***Abstract*— We study the problem of constructing controlled supermartingale functions to synthesize feedback laws that guarantee safety properties of stochastic differential equations (SDE) with control inputs. SDEs are widely used to model continuous time stochastic processes with applications ranging from financial markets to biology. In this paper, we extend classic notions from martingale theory for stochastic processes to prove that a given SDE will not exit a safe region over some finite time horizon with high probability. Our notion considers time-varying supermartingale functions that provide sharper probability bounds when compared to those that are time-independent. Furthermore, we study the controlled version of these supermartingales and the problem of synthesizing feedback control law that will maintain the state within a safe set with high probability over a given finite time horizon. We provide a projection-based algorithm for synthesizing polynomial, time-varying controlled supermartingales and corresponding feedback laws using sum-of-square (SOS) programming techniques. We implement our approach on some challenging numerical examples to demonstrate how it can synthesize control feedback laws that provide upper bounds on the probability of safety violations over a given time horizon.**

I. INTRODUCTION

Stochastic differential equations (SDEs) are models of continuous-time stochastic processes that are built on top of ordinary differential equations by adding terms based on the differential of the Brownian motion. They are naturally used to model many types of stochastic processes in areas such as control systems (e.g., modeling wind disturbances) [1], physics (e.g., modeling noise in solid state devices) [2] and finance (e.g., modeling option prices) [3]. SDEs are defined by combining drift and diffusion terms, wherein the drift term models the evolution of the states over time and the diffusion term introduces a stochastic term that is affected by the "derivative" of white noise [4]. In this paper, we study the problem of proving that given an SDE with initial conditions, its trajectories remain within a safe region for some given finite time horizon $[0, T]$ with high probability. Furthermore, given a SDE with control inputs, we study the problem of designing a feedback law that ensures bounded time safety with high probability. Tools from supermartingale theory have been used to tackle such problems [5]–[9]. However, these tools often focus on time independent supermartingales. In this paper, we focus on so-called λ-supermartingales that are time dependent through a time varying scaling term $\lambda(t)$. This allows us to prove sharper time varying bounds that tail off exponentially as time increases. Also, the computation of such supermartingales with control inputs is very challenging: leading to bilinear optimization problems that are well-known to be computationally hard. In this paper, we provide a simple iterative scheme for computing such supermartingales using sum-of-squares (SOS) programming [10], [11]. Our iterative scheme formulates a series of semidefinite programming problems that, upon convergence, yield the required supermartingales. Although convergence of our computational scheme is not guaranteed *a priori*, we demonstrate promising results on a set of challenging numerical examples that show the power of our approach to prove nontrivial bounds on probabilities of reaching unsafe sets.

Ghanbarpour and Sankaranarayanan are with the Department of Computer Science, University of Colorado Boulder, USA. Guillaume Berger is with ICTEAM institute, UCLouvain, Belgium

A. Related Work

The notion of supermartingales [12] and their use in proving properties of stochastic systems is well known [13]. Lyapunov methods have been extended to SDEs to prove stochastic stability [13]–[15]. Techniques for proving safety properties have built upon these results. Prajna et al [5] introduced the notion of stochastic barrier certificates, which serve as supermartingales to verify that stochastic hybrid systems remain within a safe set with high probability. This work laid the foundation for subsequent developments in the formal verification of probabilistic safety properties. For instance, Wisniewski and Bujorianu [8] developed a probabilistic safety analysis framework for Markov processes, further expanding the scope of barrier methods to stochastic settings. More recently, Wang et al [7] introduced the concept of stochastic control barrier functions (SCBFs) for safety-critical control of systems governed by SDEs, bridging the gap between safety verification and control design in stochastic environments Although Prajna et al demonstrate the use of polynomial supermartingales synthesized using sum-of-squares programming, the later work of Steinhardt and Tedrake [6] utilizes exponential barrier functions and semidefinite programming to derive sharper probability bounds for safety properties of discrete and continuous time stochastic systems. However, the computation of exponential supermartingales is a hard problem that also leads to bilinear constraints. In this paper, we try to prove sharp bounds on the probability of safety violations through the use of time-varying supermartingales wherein the probabilities of safety property violations are shown to decay exponentially over time. The contributions of this paper are two-fold: (a) we show how time varying supermartingale functions can provide sharp bounds; and (b) we demonstrate an approach to synthesize these functions using SOS programming.

SOS programming techniques allows us to prove properties over polynomial inequalities by encoding the problem of proving that a given polynomial is non-negative over a semi-algebraic set as a semi-definite optimization problem that can be solved efficiently [10], [11], [16]. The approach has been well integrated into packages such s the Julia SumOfSquares package [17]. Our computational approach presented here is an extension of our previous work that uses a similar approach to synthesize vector barrier function for Ordinary Differential Equations (ODEs) [18]. However, the setting of this work is entirely different,and thus, the technique has been suitably adapted.

II. Preliminaries and Problem Statement

In this section, we will first recall some classical notions of stochastic processes and define the SDE model.

A. Stochastic Process

We will briefly present some basic concepts from measure theory; further details are available elsewhere [12].

Given a set Ω (sample space), a *measurable space* is defined as a pair $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a σ-algebra on Ω. We will refer to key concepts from measure theory and stochastic processes including that of a filtration, stopping time, stopped processes and Doob's inequality. The details of these concepts are available from standard textbooks [12]. Let $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$ denote a set of "time instants".

The elements of $\mathcal{F}$ are subsets of Ω called measurable sets or events. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is known as a *probability space*. The function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is the *probability measure* defined on $(\Omega, \mathcal{F})$, assigning probabilities to events within the sample space.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a measurable space $(E, \mathcal{E})$, a random variable $X : \Omega \to E$ is a measurable function: for all $B \in \mathcal{E}$, $X^{-1}(B) \in \mathcal{F}$. A *stochastic process* is a collection of random variables $\{X_t : \Omega \to E\}_{t\in\mathcal{I}}$.

Definition 1 (Filtration): The family of σ-algebras $\{\mathcal{F}_t\}_{t\in\mathcal{I}}$ is called a *filtration* on $(\Omega, \mathcal{F})$, if for each t, $\mathcal{F}_t \subseteq \mathcal{F}$ and for $0 \leq s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$.

We will define a special class of stochastic processes called *supermartingales*.

Definition 2 (Supermartingales): Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the stochastic process $\{X_t\}_{t\in\mathcal{I}}$ is called a *supermartingale* with respect to a filtration $\mathcal{M} = \{\mathcal{M}_t\}_{t\in\mathcal{I}}$ iff the following conditions hold:

- X_t is $\mathcal{M}_t$-measurable for all $t \in \mathcal{I}$,
- $\mathbb{E}[|X_t|] < \infty$ for all $t \in \mathcal{I}$,
- $\mathbb{E}[X_t \mid \mathcal{M}_s] \leq X_s$ for all $t, s \in \mathcal{I}$, $t \geq s$.

Remark 1: In contrast to the supermartingale condition specified in Definition 2, if we have $\mathbb{E}[X_t \mid \mathcal{M}_s] \geq X_s$ for all $t, s \in \mathcal{I}$, $t \geq s$, the stochastic process is referred to as a *submartingale* with respect to the filtration $\mathcal{M}$. If a process satisfies both the submartingale and supermartingale conditions, it is called a *martingale process* [19].

Theorem 1 (Supermartingale Inequality [19]): Suppose $\{X_t\}_{t\in\mathcal{I}}$ is a real-valued supermartingale process. Then for each $c > 0$ and bounded interval $[a, b] \subset \mathcal{I}$ we have

$$c \times \mathbb{P}\left(\sup_{t\in[a,b]} X_t \geq c\right) \leq \mathbb{E}[X_a] + \mathbb{E}[\max(-X_b, 0)] , \quad (1)$$

$$c \times \mathbb{P}\left(\inf_{t\in[a,b]} X_t \leq -c\right) \leq \mathbb{E}[\max(-X_b, 0)] . \quad (2)$$

The supermartingale inequality is very useful since it bounds the probability that the supermartingale exceeds some level c inside a time interval $[a, b]$ in terms of the properties of the process at the "end points" X_a, X_b.

Definition 3 (Stopping Time): Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{M} = \{\mathcal{M}_t\}_{t\in\mathcal{I}}$, a function $\tau : \Omega \to \overline{\mathcal{I}}$ is called a stopping time with respect to $\mathcal{M}$ if for all $t \in \mathcal{I}$, we have $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{M}_t$. Informally, a stopping time τ refers to a "criterion" for stopping a process that can be evaluated at any time t by knowing the values of the X_s for $s \leq t$.

Given a stochastic process $X = \{X_t\}_{t\in\mathcal{I}}$ and a stopping time τ, the process $X_\tau = \{X_{t\wedge\tau}\}_{t\in\mathcal{I}}$ is called the stopped process, where

$$X_{t\wedge\tau}(\omega) = \begin{cases} X_t & \text{if } t \leq \tau(\omega), \\ X_{\tau(\omega)} & \text{if } t > \tau(\omega) . \end{cases}$$

If the stochastic process $X = \{X_t\}_{t\in\mathcal{I}}$ starts from a specific point $X_0 = x$, let $\mathbb{P}^x$ denote the probability measure satisfying $\mathbb{P}^x(X_0 = x) = 1$. When X_0 is a random variable with distribution μ, we denote the corresponding probability measure by $\mathbb{P}^\mu$. Likewise, we employ the notations $\mathbb{E}^x(h(X_t)) := \mathbb{E}[h(X_t)|X_0 = x]$ and $\mathbb{E}^\mu(h(X_t)) := \mathbb{E}[X_t|X_0 \sim \mu]$.

Definition 4 (Markov Property): We say that the process $\{X_t\}_{t\in\mathcal{I}}$ on a σ-algebra $(E, \mathcal{E})$ has the *Markov property* with respect to a filtration $\mathcal{M}$ if, for each $x \in E$, $D \in \mathcal{E}$, and $s, t \in \mathcal{I}$, we have $\mathbb{P}^x(X_{t+s} \in D \mid \mathcal{M}_t) = \mathbb{P}^{X_t}(X_s \in D)$.

B. Stochastic Differential Equation

Consider stochastic differential equation (SDE) as

$$dx = f(x)dt + \sigma(x)dW \quad (3)$$

where $x \in \mathbb{R}^n$ is the state of system in $\mathbb{R}^n$, W is p-dimensional Brownian motion, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the drift term, and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ is the diffusion term.

Assumption 1 (Existence and Uniqueness Assumption): The functions f and σ satisfies the following conditions (see [4, Theorem 5.2.1]):

- f, σ are measurable functions,
- (Growth Condition) For some C, and each $x \in \mathbb{R}^n$: $|f(x)| + |\sigma(x)| \leq C(1 + |x|)$,
- (Lipschitz Condition) For some L, and $x, y \in \mathbb{R}^n$: $|f(x) - f(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|$.

Remark 2: The solutions to the SDE in (3) satisfy the Markov property.

Consider the measurable space $E = \mathbb{R}^n$ equipped with the Borel measure. An observable function h is defined as $h : E \to \mathbb{R}$. Let $\mathcal{B}(E)$ be the set of all bounded measurable functions from $E = \mathbb{R}^n$ to the real line $\mathbb{R}$. The space $\mathcal{B}(E)$ forms a Banach space under the supremum norm defined as $\|h\| = \sup_{x \in E} |h(x)|$ [20].

Definition 5 (Semigroup of Operators [21]): A one-parameter family of linear operators $\{P_t\}$ which satisfies the following properties is called *semigroup*.

- $P_0 = \mathsf{Id}$,
- $P_{t+s} = P_t \circ P_s, \quad \forall s, t \geq 0$.

Definition 6 (Koopman Operator): For every $t \in \mathcal{I}$, $P_t : \mathcal{B}(E) \to \mathcal{B}(E)$ is a positive linear operator with respect to the Markov process $\{X_t\}_{t \in \mathcal{I}}$. Let $h \in \mathcal{B}(E)$ then

$$P_t h(x) = \mathbb{E}^x \left[h(X_t) \right] . \tag{4}$$

$\{P_t\}_{t \in \mathcal{I}}$ has semigroup property. Based on Chapman-Kolmogorov equation [22], for $s, t \in \mathcal{I}$ we have $P_{t+s} = P_t P_s$, and $P_0 = \mathsf{Id}$.

Definition 7 (Generator of Koopman Operator): The operator $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \to \mathcal{B}(E)$ is called the (infinitesimal) generator of the semigroup P_t, and is defined as

$$\mathcal{L}h(x) = \lim_{t \to 0} \frac{P_t h(x) - h(x)}{t}, \quad x \in E\,, \tag{5}$$

where $\mathcal{D}_{\mathcal{L}}$ be the set of all $f \in \mathcal{B}(E)$ that the the limit exists.

The Koopman generator captures the infinitesimal evolution of observables under the dynamics, similar to how the Lie derivative describes changes along trajectories in deterministic systems. The relation of the Koopman operator and its generator is expressed through Dynkin's formula.

Proposition 1: (Dynkin Formula) Let $\mathcal{I} = \mathbb{R}_{\geq 0}$. Suppose $h \in \mathcal{D}_{\mathcal{L}}$, then for $t \geq 0$ (see [23, Proposition 14.10])

$$P_t h(x) = h(x) + \mathbb{E}^x \left[\int_0^t \mathcal{L}h(X_s) ds \right] . \tag{6}$$

Remark 3: The generator of Koopman for the SDE system in (3) for a twice differentiable function h is given as

$$\mathcal{L}h(x) = \nabla h(x)^\top f(x) + \frac{1}{2} \mathsf{tr}\big((\sigma(x)\sigma(x)^\top) \times \nabla^2 h(x) \big) . \tag{7}$$

Example 1: Consider the SDE: $dx = -0.5x dt - 0.2x dW$ over $x \in \mathbb{R}$ and 1D Brownian motion $W(t)$. The drift term $a(x) = 0.5x$ and the diffusion term $\sigma(x) = 0.2x$. For the Lebesgue measurable function $h(x) = x^2$. We have $\mathcal{L}h = a(x) \cdot \nabla h + \frac{1}{2} \mathsf{tr}(\sigma \sigma^\top \nabla^2 h) = -x^2 + 0.04x^2 = -0.96x^2$.

C. Controlled SDE

In many dynamical systems, the drift term not only captures the inherent behavior of the system but may also include an additional component influenced by a control input. This control input allows for external manipulation or regulation of the system's dynamics, enabling stabilization, trajectory tracking, or optimization of performance. Consider an SDE system where the drift term includes an additional component influenced by a control input:

$$dx = (f(x) + g(x)u)dt + \sigma(x)dW\,, \tag{8}$$

where $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and the control input $u \in \mathbb{R}^m$.

Let $\mathcal{L}_u$ be the generator of (8), and $\mathcal{L}$ be the generator of the SDE without the term $g(x)u$ involving the control input. Then, we can express $\mathcal{L}_u$ in terms of $\mathcal{L}$ of SDE without control term, along with an additional term that captures the influence of the control input, as

$$\mathcal{L}_u h(x) = \mathcal{L}h(x) + \nabla h(x)^\top g(x)u\,. \tag{9}$$

D. Problem Statement

We will now present the problem statement.

Inputs: We are given a controlled SDE model according to Eq. (8) defined by functions $f(x), g(x)$ and $\sigma(x)$, a set of initial states $\mathcal{X}_0$, unsafe states $\mathcal{X}_u$ ($\mathcal{X}_0 \cap \mathcal{X}_u = \emptyset$), time horizon $T > 0$ and probablity threshold $\rho \in (0, 1)$.

Output: A continuous and differentiable control feedback law $u = \kappa(x)$ such that for all initial conditions $x_0 \in \mathcal{X}_0$, the probability that a trajectory of the resulting closed loop system starting from x_0 at time $t = 0$ reaches a state $x_t \in \mathcal{X}_u$ for $t \in [0, T]$ is less than or equal to ρ.

In this paper, we consider SDEs which are "control affine", following the form in Eq. (8), wherein the functions $f(x), g(x)$ and $\sigma(x)$ are multivariate polynomials over x. We will search for a feedback law $u = \kappa(x)$, wherein κ is a polynomial with a degree bound D. The proof of the probability bound uses the concept of a λ-supermartingale function $h(x)$, defined in Section III. Our approach, therefore, jointly synthesizes both the λ-supermartingale function given by a polynomial $h(x)$ involving the states of the system and the feedback law κ. We pose this problem as that of solving a series of sum-of-squares (SOS) optimization problems. If these problems converge onto a solution, we show that the solution yields the required feedback law κ and the λ-supermartingale function $h(x)$ that certifies the probability bounds. Section IV discusses the SOS problem formulation and properties.

III. Supermartingale and λ-Supermartingale Functions

Supermartingales are commonly used to study stochastic systems, and are particularly useful for safety, stability analysis, and stochastic control. Recall that the supermartingale inequality in Theorem 1 allows us to bound the probability of entering a given set.

Definition 8 (Supermartingale Functions): An observable function $h : E \to \mathbb{R}$ is a *supermartingale* for stochastic process $\{X_t\}$ iff $\{h(X_t)\}$ is a supermartingale process.

Lemma 1: Given the SDE system in (3), and the corresponding infinitesimal generator $\mathcal{L}$, an observable function $h \in D_{\mathcal{L}}$ is supermartingale if $\mathcal{L}h(x) \leq 0$.

Proof: To show that $\{h(X_t)\}$ is supermartingale, we need to show that $\mathbb{E}\left[h(X_t) \mid h(X_s)\right] \leq h(X_s)$ for every $t \geq s$. Using the semigroup property and Dynkin formula 1,

$$\begin{aligned}\mathbb{E}\left[h(X_t) \mid h(X_s)\right] &= P_{t-s}h(X_s) \\ &= h(X_s) + \mathbb{E}^{X_s}\left[\int_s^t \mathcal{L}h(X_r)dr\right].\end{aligned}$$

Since $\mathcal{L}h(x) \leq 0$, it follows that $\mathbb{E}\left[h(X_t) \mid h(X_s)\right] \leq h(X_s)$ thereby confirming that $\{h(X_t)\}$ is a supermartingale. ■

Example 2: The function $h(x)$ in Example 1 is a supermartingale: $\mathcal{L}h = -0.96x^2 \leq 0$ for all x.

A. λ-Supermartingales

Let $\lambda : \mathcal{I} \to \mathbb{R}$ be a continuous function over time. We introduce λ-supermartingales, which generalize the concept of supermartingales.

Definition 9 (λ-supermartingale): An observable $h : E \to \mathbb{R}$ is called λ-*supermartingale*, if there exists a continuous function $\lambda : \mathcal{I} \to \mathbb{R}$ such that $e^{-\int_0^t \lambda(s)ds}h$ is a supermartingale function.

A sufficient condition for an observable function $h \in D_{\mathcal{L}}$ to be λ-supermartingale is given in the following lemma.

Lemma 2: Given the SDE system in (3), and the corresponding infinitesimal generator $\mathcal{L}$, an observable function $h \in D_{\mathcal{L}}$ is λ-supermartingale with a continuous *and differentiable* function $\lambda : \mathcal{I} \to \mathbb{R}$ if $\mathcal{L}h \leq \lambda h$.

Proof: Since h is λ-supermartingale, the function $e^{-\int_0^t \lambda(s)ds}h$ is supermartingale. Therefore, Lemma 1 implies that $\mathcal{L}(e^{-\int_0^t \lambda(s)ds}h) \leq 0$. Hence

$$\begin{aligned}\mathcal{L}(e^{-\int_0^t \lambda(s)ds}h) &= -\lambda(t)e^{-\int_0^t \lambda(s)ds}h + e^{-\int_0^t \lambda(s)ds}\mathcal{L}h \\ &= e^{-\int_0^t \lambda(s)ds}\left(-\lambda(t)h + \mathcal{L}h\right) \leq 0\,.\end{aligned}$$

Then, since $\mathcal{L}h \leq \lambda(t)h$, h is λ-supermartingale. ■

Lemma 3: Let $h \in D_{\mathcal{L}}$ be a λ-supermartingale function with a smooth function $\lambda : \mathcal{I} \to \mathbb{R}$, then for all $x \in \mathbb{R}^n$,

$$P_t h(x) \leq e^{\int_0^t \lambda(s)ds}h(x)\,. \tag{10}$$

Proof: Using that the semigroup operator is defined as $P_t = e^{t\mathcal{L}}$, for each fixed x, taking the time derivative of $P_t h$ we have that $\frac{d}{dt}P_t h(x) = \mathcal{L}e^{t\mathcal{L}}h(x)$. Using that semigroup operator commutes with infinitesimal operator, we have $\frac{d}{dt}P_t h(x) = e^{t\mathcal{L}}\mathcal{L}h(x) \leq e^{t\mathcal{L}}\lambda(t)h(x) = \lambda(t)P_t h(x)$. Then, we have $P_t h(x) \leq e^{\int_0^t \lambda(s)ds}h(x)$. ■

B. Probabilistic Forward Invariance and Safety Inference using Supermartingale Functions

Here, we explore forward invariance and safety probabilities using supermartingale functions.

Definition 10 (First Hitting Time): Suppose $D \in \mathcal{E}$ and $\omega \in \Omega$. The *first hitting time* τ_D of the process $X = \{X_t\}_{t \in \mathcal{I}}$ to the set D is a random variable defined as

$$\tau_D(\omega) := \inf\{t \in \mathcal{I} : X_t(\omega) \in D\}.$$

Note that $\tau_D(\omega) = \infty$ if $X_t(\omega) \notin D$ for all $t \in \mathcal{I}$. Similarly, the *first exit time* from a set D is defined as the first hitting time of D^c (the complement of the set D).

Let $E = \mathbb{R}^n$, $\mathcal{I} = \mathbb{R}_{\geq 0}$, $h \in \mathcal{D}_{\mathcal{L}}$ be a nonnegative supermartingale function, and O_c be the super-level set of h for $c > 0$:

$$O_c := \{x \in \mathbb{R}^n \mid h(x) \geq c\}\,. \tag{11}$$

Suppose τ_{O_c} indicates the first hitting time of the solution to the SDE system (3) starting from $x \in \mathbb{R}^n \backslash O_c$ to the set O_c. The probability of hitting O_c within the time interval $[0, T]$ starting from x, is formulated as

$$\mathbb{P}^x(\tau_{O_c} \leq T) = \mathbb{P}^x\big(\sup_{t \in [0,T]} h(X_t) \geq c\big)\,. \tag{12}$$

Let us first consider a bound for a fixed time $t \in \mathcal{I}$. Let $h \in \mathcal{D}_{\mathcal{L}}$ be a nonnegative λ-supermartingale with smooth function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and let O_c in (11) be a super-level set of h for $c > 0$. The probability that X_t enters the set O_c can be upper bounded directly using Markov inequality follows:

$$\left.\begin{aligned}\mathbb{P}^x(h(X_t) \geq c) \quad &\leq \tfrac{1}{c}\mathbb{E}^x[h(X_t)] \;=\; \tfrac{1}{c}P_t h(x) \\ &\leq \tfrac{1}{c}\exp\left(\int_0^t \lambda(s)ds\right)h(x)\end{aligned}\right\} \tag{13}$$

Note that the bound above holds for a specific time t. Our goal is to bound the probability over a time interval $[0, T]$. The Lemma below provides us with a required bound. For convenience, let $\Lambda(t) = \int_0^t \lambda(s)ds$.

Lemma 4: Given the SDE system in (3), and the corresponding infinitesimal generator $\mathcal{L}$. Let $h \in D_{\mathcal{L}}$ be a nonnegative λ-supermartingale function with the continuous function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Let O_c be as defined (11) and $S \subseteq \mathbb{R}^n$ be such that $S \cap O_c = \emptyset$. For $x \in S$,

$$\mathbb{P}^x\big(\tau_{O_c} \leq T\big) \leq \frac{1}{c}\left(\sup_{t \in [0,T]} e^{\Lambda(t)}\right)h(x)\,. \tag{14}$$

If we have initial distribution μ with support in S, then

$$\mathbb{P}^\mu\big(\tau_{O_c} \leq T\big) \leq \frac{1}{c}\left(\sup_{t \in [0,T]} e^{\Lambda(t)}\right)\mathbb{E}^\mu[h(X_0)]\,. \tag{15}$$

Proof: Let t range over the interval $[0, T]$. We have

$$\mathbb{P}(\exists t : h(X_t) \geq c) = \mathbb{P}\left(\exists t : e^{-\Lambda(t)}h(X_t) \geq ce^{-\Lambda(t)}\right)$$

$$= \mathbb{P}\left(\sup_t e^{-\Lambda(t)} h(X_t) \geq \inf_t c e^{-\Lambda(t)}\right)$$
$$\leq \frac{\mathbb{E}[h(X_0)]}{\inf_t c e^{-\Lambda(t)}} = \frac{1}{c}\left(\sup_t e^{\Lambda(t)}\right)\mathbb{E}[h(X_0)]$$

If X_0 is a Dirac distribution at x, then $\mathbb{E}[h(X_0)] = h(x)$, proving (14). If $X_0 \sim \mu$, then we establish (15). ■

Remark 4: A supermartingale is, in fact, a λ supermartingale with $\lambda = 0$. The term $\left(\sup_{t\in[0,T]} e^{\Lambda(t)}\right)$ in Eq. (14) is advantageous if $\lambda(t) \leq 0$ for all $t \in [0, T]$. First, the probability bound $\mathbb{P}^x(h(X_t) \geq c)$ decreases exponentially over time t yielding a tight bound for a specific time instant t. This is illustrated through the numerical examples in Section V. For instance the plots in Figs. 1, 2 and 3 show how the probability bounds for reaching an unsafe set tails off over time, thanks to the $\lambda(t)$ term. Furthermore, $\sup_{t\in[0,T]} e^{\Lambda(t)} \leq e^{\Lambda(0)} = 1$. This latter fact is useful in computing bounds that hold over a time horizon $[0, T]$.

We will now consider the case for controlled SDE (8).

Lemma 5: Given the controlled affine SDE in (8), let $h \in D_{\mathcal{L}}$ be a nonnegative λ-supermartingale function with the continuous function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Suppose $S \subset \mathbb{R}^n$ and $h(x) \leq c$ for $x \in S$. Let $O_c = \{x \in \mathcal{X} : h(x) \geq c\}$ be a super-level set of h for $c > 0$. If there exists a differentiable control law $u = \kappa(x)$ such that for each $x \in S$:

$$\mathcal{L}_u h(x) = \mathcal{L}h(x) + \nabla h(x)^\top g(x) u(x) \leq \lambda h(x)\,, \quad (16)$$

then, the probability of reaching O_c at time t starting from $x \in S$ is bounded as:

$$\mathbb{P}\left(h(X_t) \geq c\right) \leq \frac{1}{c} e^{\Lambda(t)} \mathbb{E}[h(X_0)]\,.$$

Likewise, if $t \in [0, T]$, then

$$\mathbb{P}\left(\exists\ t : h(X_t) \geq c\right) \leq \frac{1}{c}\left(\sup_t e^{\Lambda(t)}\right)\mathbb{E}[h(X_0)]\,.$$

The proof directly follows by applying Lemma 5 to the closed-loop SDE obtained by plugging-in $u = \kappa(x)$. Returning to the overall problem statement in Section II-D, we seek a λ-supermartingale $h(x)$ and feedback law κ that satisfy the following criteria:

$$\left.\begin{array}{l} h(x) \geq 0,\ \forall x \in \mathcal{X};\quad \lambda(t) \leq 0,\ \forall t \in [0, T];\\ h(x) \leq \rho,\ \forall x \in \mathcal{X}_0;\quad h(x) \geq 1,\ \forall x \in \mathcal{X}_u;\\ \mathcal{L}h(x) + \nabla h(x)^\top g(x)\kappa(x) \leq \lambda(t) h(x),\\ \qquad\qquad\qquad\qquad \forall x \in \mathcal{X}, t \in [0, T]\,. \end{array}\right\} \quad (17)$$

Theorem 2: Let $h(x), \kappa(x)$ satisfy (17) over $x \in \mathcal{X}$ and $t \in [0, T]$ for given controlled SDE involving f, g, σ, sets $\mathcal{X}_0, \mathcal{X}_u$, time horizon $T > 0$ and probability threshold $\rho \in (0, 1)$. The controlled SDE (8) under the feedback $u = \kappa(x)$ satisfies $\mathbb{P}\left(\tau_{X_u} \leq T\right) \leq \rho$.

Proof: (Sketch) The proof is a direct application of Lemma 4 noting that $\mathcal{X}_u \subseteq O_c$ for $c = 1$. Also, $h(X_0) \leq \rho$. Furthermore, $\sup_t e^{\Lambda(t)} = 1$ since $\lambda(t) \leq 0$ for all $t \in [0, T]$. ■

IV. Synthesis of Supermartingales

In this section, we use an iterative approach to synthesize polynomial λ-supermartingale functions and control feedback laws for a given SDE. We will restrict our attention to nonlinear SDEs whose drift and diffusion functions are described by polynomials over x, t. Furthermore, we will synthesize λ-supermartingales $h(x)$, wherein $\lambda(t)$ and $h(x)$ are polynomials over t, x, respectively.

Let $\mathbb{R}[x]$ denote the set of all polynomials for $x \in \mathbb{R}^n$. We assume that the SDE system in (8) is polynomial, the functions f, g, σ are polynomials.

Definition 11 (Sum-of-Squares): A polynomial $g \in \mathbb{R}[x]$ is a *sum-of-squares* if $g = \sum_{i=1}^{r} p_i^2$ for $r \in \mathbb{N}$ and $p_i \in \mathbb{R}[x]$. The set of SOS polynomials is denoted by $\mathrm{SOS}[x]$.

We use monomial basis functions to express each polynomial as the inner product of a coefficient vector with a monomial vector. Suppose $\phi_d(x)$ denotes the monomial vector that contains all monomials over x with degree $\leq d$. Then, each polynomial $p(x)$ of maximum degree d can be expressed as $p(x) = v^\top \phi_d$ where $v \in \mathbb{R}^{r_d}$ is the coefficient vector and $r_d = \binom{n+d}{n}$. We define the (coefficient) distance between two polynomials $h_i(x) = v_i^\top \phi_d$ and $h_j(x) = v_j^\top \phi_d$ as

$$\mathrm{d}(h_i, h_j) := \|v_i - v_j\|_2\,. \quad (18)$$

First, we consider polynomial SDEs without control inputs. Our goal is to synthesize a λ-supermartingale function $h(x) \in \mathbb{R}[x]$ and $\lambda(t) \in \mathbb{R}[t]$ that satisfies the condition:

$$\forall\ x \in \mathcal{X} \subset \mathbb{R}^n,\ \mathcal{L}(h(x)) \leq \lambda h(x) \quad (19)$$

Let d be an *a priori* fixed degree bound for $h(x)$. We write $h(x) = v_h^\top \phi_d$ for an unknown vector $v_h \in \mathbb{R}^{r_d}$. We do the same for λ as a function of t, $\lambda(t) = \omega_\lambda^\top \psi_q(t)$, where $\psi_q(t)$ is a basis of monomials up to some degree q in t.

Since $\mathcal{L}$ is a linear operator, we have that $\mathcal{L}h(x) = \mathcal{L}\left(v_h^\top \phi_d\right) = v_h^\top \mathcal{L}\phi_d$. Note that $\mathcal{L}\phi_d$ is a vector of polynomials whose degrees can exceed d. We relax condition (19) using the sum-of-squares (SOS) approach [10], [11], [16]:

$$\left.\begin{array}{l} \textsf{find } v_h, \omega_\lambda \textsf{ s.t.}\\ (\omega_\lambda^\top \psi_q(t)) \times (v_h^\top \phi_d(x)) - (v_h^\top \mathcal{L}\phi_d(x)) \textsf{ is SOS} \end{array}\right\} \quad (20)$$

Note that the constraints are refined further using Putinar/Schmugden's positivstellensatz to encode the restriction that $t \in [0, T]$ along with restrictions to the state-space x that are stated as polynomial inequalities [10], [11], [16]. Also, if we require h to be positive semi-definite, we can add the extra SOS constraint that $h(x) = v_h^\top \phi_d(x)$ is SOS. We will omit these to simplify the presentation, but include them in our implementation. The SOS constraints are translated into semi-definite programming problems. In this case, however, the constraints obtained are *bilinear*, since they involve a product of the unknowns v_h and ω_λ.

The resulting feasibility problem is therefore non-convex and hard to solve, in general [24].

Remark 5: A common approach to solving bilinear constraints uses the idea of "alternating co-ordinate descent" that goes by different names in the control theory and optimization literature [25], [26]. Here, the idea is to iterate starting from initial guesses $\omega_\lambda^{(0)}, v_h^{(0)}$ provided by the user. At each iteration, $\omega_\lambda^{(i+1)}$ is obtained by solving (20) by fixing $v_h = v_h^{(i)}$ and likewise, $v_h^{(i+1)}$ is obtained by fixing $\omega_\lambda = \omega_\lambda^{(i+1)}$. Doing so, yields a sequence of convex semi-definite programming problems. We note that such a scheme has the possibility of converging to a local "saddle point" and has poor performance in practice [27].

Inspired by Alternating Minimization Algorithm [28, Section 1.6.3], we propose a projection-based approach that allows all the decision variables v_h, ω_λ to potentially change during each iteration but nevertheless yields convex optimization problems. For simplicity, we write $v = v_h$ and $\omega = \omega_\lambda$. Let h_0 be an initial *guess*, $h_0 = v_0^\top \phi_d$. The proposed iterative scheme solves a the following optimization problem to find $h_{i+1} = v_{i+1}^\top \phi_d$ given $h_i = v_i^\top \phi_d$ for $i \geq 0$:

$$\left.\begin{array}{ll} \min\limits_{v,\omega} & \|v - v_i\|_2 \\ \mathsf{s.t.} & (v_i^\top \phi_d(x)) \times (\omega^\top \psi_q(t)) - v^\top \mathcal{L}\Phi_d \text{ is SOS} \\ & \text{Additional constraints over } v, \omega \end{array}\right\} \tag{21}$$

Eq. (21) computes (v_{i+1}, ω_{i+1}) as a function of v_i. Note that ω_i is not used.

Lemma 6: At any iteration i, if the optimization problem (21) has an optimal value of 0, then $v_{i+1} = v_i$ and the values $(v_h, \omega_\lambda) = (v_{i+1}, \omega_{i+1})$ satisfy Eq. (20).

Algorithm 1 summarizes the overall approach. The algorithm constructs a sequence of iterates starting from an initial guess v_0 and iterating until a limit N is reached. Rather than terminate when the optimal value from solving Eq. (20) is precisely 0, we will do so when it falls below some tolerance. Finally, the approach checks that Eq. (20) holds on the final result, failing with the result is rejected. In practice, we find that multiple trials using variations on the initial guess h_0 can succeed in discovering suitable functions $h(x)$.

Controlled Supermartingale Synthesis: We will now extend our approach to controlled supermartingale synthesis. Consider a controlled SDE as specified in (8). For each monomial in our basis $m(x) \in \phi_d$, we can write: $\mathcal{L}_u m(x) = \mathcal{L}m(x) + (\nabla m(x))^\top g(x) u$. Suppose we parameterize the feedback $u = \kappa(x)$ as a polynomial function $u = K\phi_d$ with an unknown matrix of coefficients $K \in \mathbb{R}^{m \times r_d}$ and over the same monomial basis ϕ_d (for convenience). The approach of alternating projection can be extended wherein at each iteration we update a triple (v_i, ω_i, K_i) starting with some

Data: *initial guess:* v_0, *monomial basis:* $\phi_d(x), \psi_q(t)$, *tolerance parameter:* $\epsilon > 0$, *iteration limit:* N
Result: SUCCESS with $(v_h, \omega_\lambda$ satisfying Eq. (20) or FAILURE conditions.
for $i = 0, \ldots, N$ **do**
    Solve SOS problem (21). Let o be the optimal objective and v_{i+1}, ω_{i+1} be the results.;
    if *($o \leq \epsilon$)* **then**
        Check if $v_h = v_{i+1}, \omega_\lambda = \omega_{i+1}$ satisfy Eq. (20). ;
        if *check succeeded* **then**
            return (v_h, ω_λ) ;
        else
            return FAILURE;
        end
    end
end
return FAILURE;

Algorithm 1: Algorithm for computing λ supermartingales using iterated projection approach.

initial guess v_0, K_0.

$$\begin{aligned} \min_{v,w,K} \quad & \|v - v_i\| + \|K - K_i\| \\ \text{s.t. } & \big(v_i^\top \phi_d(x)\big) \times \big(\omega^\top \psi_q(t)\big) - \big(v^\top \mathcal{L}\phi_d(x)\big) \\ & \quad + \big(v_i^\top \nabla \phi_d(x)\big) \times g(x) \times (K\phi_d(x)) \ \text{ is SOS}\,, \\ & \text{Additional constraints over } v, w, K\,. \end{aligned} \tag{22}$$

Once again, we can apply a modified version of the iterative scheme given a controlled SDE: (a) We initialize using a guess (v_0, K_0); (b) at each iteration, we update $(v_{i+1}, \omega_{i+1}, K_{i+1})$ by solving Eq. (22); and (c) if we achieve convergence, we apply Eq. (22) to check our final result.

Finally, we conclude by focusing on the solution to the problem statement in Section II-D. Given initial set $\mathcal{X}_0$, unsafe set $\mathcal{X}_u$, time horizon T and probability bounds ρ, we find a solution to Eq. (17) by solving the following SOS program iteratively.

$$\left.\begin{array}{ll} \min\limits_{v,w,K} & \|v - v_i\| + \|K - K_i\| \text{ s.t.} \\ \mathsf{C1}: & v_i^\top \phi_d(x) \text{ is SOS on } x \in \mathcal{X}\,, \\ \mathsf{C2}: & -w^\top \psi_q(t) \text{ is SOS for } t \in [0,T]\,, \\ \mathsf{C3}: & v_i^\top \phi_d(x) \leq \rho \text{ for } x \in \mathcal{X}_0\,, \\ \mathsf{C4}: & v_i^\top \phi_d(x) \geq 1 \text{ for } x \in \mathcal{X}_u\,, \\ \mathsf{C5}: & \big(v_i^\top \phi_d\big) \times \big(w^\top \psi_q\big) - v^\top \mathcal{L}\phi_d \\ & \quad + \big(v_i^\top \nabla \phi_d\big) \times g(x) \times (K\phi_d) \\ & \qquad \text{is SOS } (t,x) \in [0,T] \times \mathcal{X}\,. \end{array}\right\} \tag{23}$$

The constraint C1 imposes non-negativity of $h(x)$, C2 enforces the negativity of $\lambda(t)$ over $[0,T]$, C3, C4 places limits for $h(x)$ on the initial set $\mathcal{X}_0$ and unsafe set $\mathcal{X}_u$, C5 denotes the λ supermartingale property.

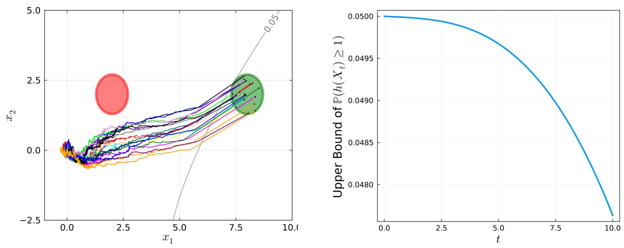

Fig. 1: **(Left)** Twenty randomly sampled trajectories of the closed loop synthesized for Example 3 and **(Right)** upper bound of probability of reaching unsafe set obtained by applying Eq. (13) .

V. Experimental Results

We now present the results of our approach on a few selected numerical examples of SDEs. In each case, we solve an instance of the problem stated in Section II-D for various controlled SDEs with given $\mathcal{X}_0, \mathcal{X}_u$, threshold T and probability bounds ρ using the iterative algorithm from Section IV and the constraints shown in (23). Our implementation uses the Julia programming language and the `SumOfSquares.jl` package of Legat et al [17]. Our implementation and the results of the numerical examples will be made available through our GitHub repository [1].

Example 3: Consider the following controlled SDE

$$dx = (f(x) + g(x)u)dt + \sigma(x)dW, \tag{24}$$

where $f(x) = \begin{pmatrix} -x_1^3 + 0.5x_2 \\ -x_1 - 2.0x_2 \end{pmatrix}$, $g(x) = I_{2\times 2}$, and $\sigma(x) = \begin{pmatrix} 0.2x_1 & 0.1x_2 \\ -0.1(x_1+x_2) & 0.2 \end{pmatrix}$. Suppose the initial set is $\mathcal{X}_0 = \{x \in \mathbb{R}^2 \,:\, (x_1-8)^2 + (x_2-2)^2 \leq 0.5\}$, and the unsafe set is $\mathcal{X}_u = \{x \in \mathbb{R}^2 \,:\, (x_1-2)^2 + (x_2-2)^2 \leq 0.5\}$. Applying the iterative algorithm to the problem in (23), the feedback control law and the λ-supermartingale observable function h are generated for $T = 10$, and $1-\rho = 0.95$. Figure 1 (left) shows twenty trajectories generated from various initial states $\mathcal{X}_0$. The probability of reaching the super-level set of h at each t calculated using Eq. (13) (i.e. $\mathbb{P}(h(X_t) \geq 1)$) as a function of time t is shown in Figure 1 (right).

The maximum degree d of monomials $\phi_d(x)$ was taken to be 8, and that for $\lambda(t)$ was 2. The stopping threshold for the algorithm is $\epsilon = 10^{-10}$.

Example 4: Consider a controlled Lotka Volterra system with $f(x) = \begin{pmatrix} 1.5x_1 - x_1x_2 \\ -x_2 + 1.2x_1x_2 \end{pmatrix}$, $g = I$, and $\sigma = \begin{pmatrix} 0.2x_1 & 0 \\ 0 & 0.2x_2 \end{pmatrix}$. Suppose $\mathcal{X}_0$, $\mathcal{X}_u$ are given as

$$\mathcal{X}_0 = [0.5, 2]^2\,,\ \mathcal{X}_u \quad = \{x \in \mathbb{R}^2 \,:\, x_1 + x_2 \geq 5\}\,.$$

Applying the iterative approach to the problem in (23), the feedback control law and the λ-supermartingale observable function h are generated for $T = 10$, and $1-\rho = 0.95$. Figures 2(left) show the normalized feedback law, while Figure 2 (right) plots the bounds on probability for specific time instants. The maximum degree for $\phi_d(x)$ was taken to be 8 and that for $\psi_q(t)$ was taken to be 2. ϵ was set to 10^{-8}.

[1] `https://github.com/MasoumehGHM/timedKoopmanScalar`

Example 5: Consider the approximate pendulum SDE system with $f(x) = \begin{pmatrix} x_2 \\ -x_1 - x_1^3 \end{pmatrix}$, $g = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$. Suppose $\mathcal{X}_0$, $\mathcal{X}_u$ are given as

$$\mathcal{X}_0 = \ [-\frac{\pi}{8}, \frac{\pi}{8}] \times [-0.2, 0.2],$$

$$\mathcal{X}_u = \ \left\{ x \in \mathbb{R}^2 \,:\, |x_1| \geq \frac{7\pi}{8},\, |x_2| \geq 5 \right\}.$$

The feedback control law and the λ-supermartingale observable function h are generated for $T = 5$, and $1-\rho = 0.75$. Figure 3 (left) shows the trajectories of twenty random generated initial points from $\mathcal{X}_0$. Figure 3(right) illustrates probability bounds at various time instants. The maximum degree of polynomials in x was taken to be 8, and the maximum degree for λ was 2. The stopping threshold for the algorithm was set to $\epsilon = 10^{-10}$.

VI. Conclusion

Thus, our proposed approach effectively leverages supermartingale functions to compute feedback laws that guarantee probabilistic safety properties of SDE models in the presence of control inputs. Future work will aim to extend this methodology to accommodate multiple observable functions, extending to stochastic jump differential equations and stochastic hybrid systems.

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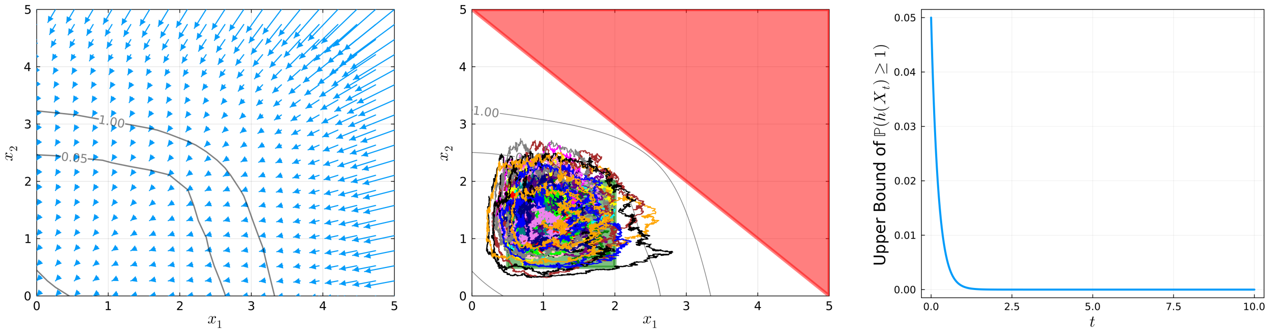

Fig. 2: **(Left)** Closed loop feedback control law of Example 4 and the $0.05, 1$-level sets of the supermartingale function; **(Middle)** Twenty trajectories of Example 4 from randomly generated initial points in $\mathcal{X}_0$, the red shadowed area indicates the unsafe set; and **(Right)** upper bound of probability of reaching unsafe set as a function of time.

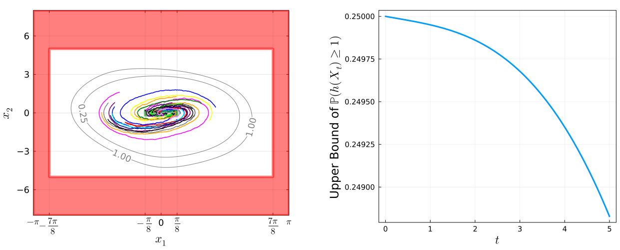

Fig. 3: **(Left)** Twenty trajectories of Example 5 from randomly generated initial points in $\mathcal{X}_0$, the red shadowed area indicates the unsafe set; and **(Right)** upper bound of probability of reaching unsafe set as a function of time.

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