Exercises on reductions
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Definition 1 (Many-one reduction)**.** We say that A *polynomial-time many-one reduces* to B, sometimes called *Karp reduces*, denoted $A \leq_m^p B$, if there is a polynomial-time computable function r such that for all strings x,

$$x \in A \Leftrightarrow r(x) \in B$$

1. Show that $\leq_m^p$ is transitive: if $A \leq_m^p B$ and $B \leq_m^p C$, then $A \leq_m^p C$.

2. We say that a complexity class $\mathcal{C}$ is *closed under polynomial-time many-one reductions* if $B \in \mathcal{C}$ and $A \leq_m^p B$ implies $A \in \mathcal{C}$.

   (a) Prove that P, NP, and coNP are closed under polynomial-time many-one reductions.

   (b) Prove that the previous result relativizes, that is, for any oracle X, P^X NP^X, and coNP^X are closed under polynomial-time many-one reductions.

   (c) Conclude that for all k, $\mathsf{\Sigma_k P}$ and $\mathsf{\Pi_k P}$ are closed under $\leq_m^p$, and also that PH is closed under $\leq_m^p$.

3. Show that EXP and PSPACE are closed under $\leq_m^p$. Show that this result relativizes.

4. E denotes the class of languages decidable in time $2^{O(n)}$, sometimes called "simply exponential time" (to distinguish it from $2^{\mathrm{poly}(n)}$).

   (a) Show that E is a proper subset of EXP, and that the closure of EXP is the closure of E under $\leq_m^p$ (that is, EXP is the smallest class containing E and closed under $\leq_m^p$).

(b) Show that E is closed under polynomial-time many-one reductions *with linear stretch.* What this means is that there is a polynomial-time many-one reduction r and a constant c such that $|r(x)| \leq c|x|$ for all strings x.

(c) Show that the previous two parts relativize.

Definition 2. We say that A *polynomial-time Turing reduces* to B (sometimes called *Cook reduces*), denoted $A \leq_T^p B$, if there is a polynomial-time oracle TM $M^\square$ such that A is correctly decided by M^B.

3. (a) Show that $\leq_T^p$ is transitive.

   (b) Show that for any oracle X, we have $\mathsf{P}^{(\mathsf{P}^X)} = \mathsf{P}^X$. (Realize that this is the same question as part (a).)

4. Prove that P is closed under $\leq_T^p$.

5. Is NP closed under $\leq_T^p$? What happens (complexity-class-wise) if it is? What about $\Sigma_\mathsf{k}\mathsf{P}$?

6. Prove that PH is closed under $\leq_T^p$.

7. Prove that EXP and PSPACE are closed under $\leq_T^p$.

8. Prove that if L is complete for a complexity class $\mathcal{C}$ under $\leq_T^p$ reductions, then $\mathsf{P}^\mathcal{C} = \mathsf{P}^L$. Conclude that $\mathsf{P}^{\mathsf{NP}} = \mathsf{P}^{\mathsf{coNP}} = \mathsf{P}^{SAT}$.

Definition 3. We say that A *polynomial-time truth-table reduces* to B denoted $A \leq_{tt}^p B$ if there are polynomial-time functions q and V such that

- $q(x)$ outputs a tuple of strings $q(x) = (q_1, \ldots, q_k)$ ("q" for "queries")
- $x \in A$ if and only if $V(x, B(q_1), \ldots, B(q_k)) = 1$.

Note that the number of queries k can depend on x.

We say that A *polynomial-time k-truth-table reduces* to B, denoted $A \leq_{k-tt}^p B$, if $A \leq_{tt}^p B$ and the number of queries made in the reduction is at most k. (In particular, $\leq_{tt}^p$ is the same as $\leq_{poly-tt}^p$.)

9. (a) Prove that $\leq_{tt}^p$ is transitive.

   (b) Prove that $\leq_{1-tt}^p$ is transitive.

   (c) Prove that $A \leq_{k-tt}^p B \leq_{\ell-tt}^p C$ implies $A \leq_{k\ell-tt}^p C$.

10. Prove that

$$A \leq_m^p B \Rightarrow A \leq_{1-tt}^p B \Rightarrow A \leq_{2-tt}^p B \Rightarrow \cdots \Rightarrow A \leq_{tt}^p B \Rightarrow A \leq_T^p B.$$

11. Prove that P, EXP, and PSPACE are closed under $\leq_{k-tt}^p$ and $\leq_{tt}^p$.

12. Is NP closed under $\leq_{1-tt}^p$? What happens if it is?

13. Prove that PH is closed under $\leq_{tt}^p$.