A Combinatoric Interpretation of Dual Variables for Weighted Matching Problems

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Abstract

We consider four weighted matching-type problems: the bipartite graph and general graph versions of matching and f-factors. The linear program duals for these problems are shown to be weights of certain subgraphs. Specifically the so-called y duals are the weights of certain maximum matchings or f-factors; z duals (used for general graphs) are the weights of certain 2-factors or $2f$-factors. The y duals are canonical in a well-defined sense; z duals are canonical for matching and more generally for b-matchings (a special case of f-factors) but for f-factors their support can vary. As weights of combinatorial objects the duals are integral for given integral edge weights, and so they give new proofs that the linear programs for these problems are TDI.

1 Introduction

The weighted versions of combinatorial problems, e.g., weighted bipartite matching, minimum cost network flow, are usually studied by introducing linear programming duals to ideas developed in the cardinality version of the problem (e.g., augmenting paths for many problems [7]; blossoms in [5] and [4] for matching, etc.), This paper takes a closer look at the combinatorics of the weighted problems without reference to the unweighted cardinality version. We present combinatoric interpretations of the linear programming duals, that are in some sense canonical. These duals give simple proofs of classic minimax theorems for cardinality problems.

The results are easily summarized. We study weighted versions of bipartite and general graph matching, and bipartite and general graph f-factors and b-matchings. We first summarize general graph matching.

Recall that a graph G is *critical* for general graph matching if every subgraph $G-v$, $v \in V$, has a perfect matching. Consider such a critical graph.

The dual variable for a vertex, $y(v)$, is the weight of a maximum perfect matching on $G-v$.

The "z" duals are derived from subgraphs we call 2-unifactors. A 2-unifactor is a perfect 2-matching with precisely one odd cycle. Take the maximum weight 2-unifactor, contract its cycle, and repeat the process until the graph shrinks to one vertex. Each z dual is difference in weight between 2 consecutive unifactors. In fact these unifactors also give the value of y duals, by "splitting" the unifactor into two halves.

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These duals are canonical, i.e., essentially unique (unlike the usual duals). This is clear for the y duals, and we prove it for the z duals if we add the desirable property of laminarity.

The duals are easily translated to optimum duals for maximum weight perfect matching. It is well-known that other versions of weighted matching (e.g., maximum weight matching) are easily translated to maximum perfect matching, so we do not consider them.

The duals are weights of combinatorial objects and so have integral value when the weight function is integral. Thus our duals give alternate proofs that the linear programming formulation of general graph matching is TDI [13].

The duals give a simple proof of Edmonds' odd set cover theorem for maximum cardinality matching [7], which we omit but see below. The duals can be used to simplify the correctness proof for the matching algorithm of Micali and Vazirani [11], again omitted (y duals give shortest augmenting path lengths, z duals give their "tenacity").

Our other problems are treated in a similar fashion with similar results. Rather than repeat them we just mention the differences or other cogent points. The bipartite problems only have y duals, and their interpretation is similar to the above. Although no bipartite graph is critical in the above sense, we use a minor variation. Of course we show the same extension to maximum weight perfect matching and f-factors. We illustrate proofs of cardinality minimax theorems (as mentioned above) by sketching how the duals for bipartite f-factors lead to a proof the max-flow min-cut theorem.

For general graph f-factors the same description of the duals applies,[1] using $2f$-unifactors. A $2f$-unifactor is a $2f$-factor, in the graph where each original multiplicity has been doubled, that has precisely one odd cycle.[2] Although this definition is the exact analog of 2-unifactors, values of f larger than 1 introduce new topological features that make f-factors the most involved part of our paper. For f-factors the y duals are canonical, but an example shows optimum z duals can have different support. However for the special case of b-matchings both y and z are canonical.

Our work is most closely related to views of general graph matching that either avoid blossoms, e.g. [1, 2, 3], or study their structure [14]. For f-factors various derivations of the b-matching polytope and its relatives, and its integrality properties, are given and referenced in [13, Ch.31–32].

Sections 2–5 discuss bipartite matching, bipartite f-factors, general matching and general f-factors (along with general b-matchings) respectively. We could use strictly analogous terminology and notions in all of these sections, but we prefer to take advantage of the special features of each setting. Most notably, Section 4 does not make explicit use of 2-unifactors, instead we use "ζ values". The reader will recognize the obvious correspondence to 2-unifactors when $2f$-unifactors are defined in Section 5. Each section begins by reviewing the linear program duals for the problem at hand. The last Section 6 gives some promising directions for future work. The Appendix gives an algorithm for finding a maximum f-factor from the "blossom tree" representation defined in Section 5.

We close this section with some notation and terminology. The symmetric difference of sets S and T is denoted $S \oplus T$. If e is an element then in set expressions we allow e to represent the singleton set $\{e\}$, e.g., $S \oplus e$ denotes $S \oplus \{e\}$, $S + e$ denotes $S \cup \{e\}$. If f and g are real-valued functions on elements and S is a set of such elements, $f(S)$ denotes $\sum\{f(v) : v \in S\}$ and $fg(S)$ dentoes $\sum\{f(v)g(v) : v \in S\}$.

We consider undirected graphs and multigraphs. For the latter loops vv are allowed. Also the notation uv refers to any edge joining u and v; the context will indicate that the choice of uv remains fixed over the current argument. Two edges joining u and v are distinct objects; thus for

[1] Definitions of criticality for f-factors that differ from ours are given in [10], also [13, p.559].

[2] In some contractions the "cycle" can include self-loops on vertices.

instance $\{uv\}$ is the set of all edges joining u and v (not a multiset). Note a loop vv is one edge and contributes two to the degree of v. For a multigraph, when a set of vertices C is contracted to a vertex $\overline{C}$, $\overline{C}$ has a loop unless C is independent; also parallel edges are retained.

A *walk* is a sequence $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices v_i and edges e_i, with $e_i = v_{i-1}v_i$. We say v_0v_k-*walk* to indicate the two endpoints. The *length* of the walk is k, and the parity of k makes the walk *even* or *odd*. A *trail* is an edge-simple walk. For multigraphs a trail can contain parallel edges since they are distinct. A *circuit* is a trail that starts and ends at the same vertex. The vertex-simple analogs are *path* and *cycle*, respectively. In a multigraph a loop is considered to be a simple cycle. If a path P contains vertices i, j then $P(i,j)$ denotes the subpath from i to j.

For any subgraph H (e.g., a path), $V(H)$ ($E(H)$) denotes its set of vertices (edges). For convenience we often consider a subgraph to be its vertex set or edge set, when context makes clear which is meant. For example if P is a path with last edge vt then $P - vt + vx$ is a path ending at x instead of t.

For a set of vertices S, $\delta(S)(\gamma(S))$ is the set of edges with exactly one (two) vertices in S. If the graph for these sets is unclear we include it as an argument, e.g., for a subgraph H, $\delta(S, H)$. For a subgraph H, $d(v, H)$ denotes the degree of vertex v in H.

A *matching* is a set of edges, no two of which share a vertex. A *free* vertex is not on any matched edge. A *perfect matching* has no free vertices. For any vertex v a v-*matching* is a perfect matching on $G - v$. G is *critical* (or *hypomatchable*) if it has a v-matching for every v. An *alternating path (cycle)* is a simple path (cycle) whose edges are alternately matched and unmatched; an *augmenting path* is an alternating path joining two distinct free vertices. Suppose each edge e of G has a real-valued weight $w(e)$. A *maximum perfect matching* M is a perfect matching of largest possible total weight (by our conventions this weight is $w(M)$). A *maximum* v-*matching* is defined similarly. If P is an alternating path for a matching M, its *weight with respect to* M is $w(P, M) = w(P \oplus M) - w(M) = w(P - M) - w(P \cap M)$.

Now let G be a multigraph. Parallel edges and loops are allowed. Let $f : V \to \mathbb{Z}_+$ be a "degree-constraint" function. $\mathbb{Z}_+$ denotes the set of nonnegative integers. An f-*factor* is a subgraph H where each vertex v has $d(v, H) = f(v)$. Other terminology for f-factors follows by analogy with matching, e.g., when G has edge weights, a *maximum* f-*factor* has the greatest weight possible. Note that parallel edges uv can have distinct weights $w(uv)$.

2 Bipartite matching

This section interprets the dual variables for weighted bipartite matching as weights of matchings. Later on we do the same for f-factors and general graphs. In all cases the dual problem is first reviewed and then the interpretation is derived.

Consider a bipartite graph G with vertex sets V_0, V_1, edge set E and weight function $w : E \to \mathbb{R}$. Recall the linear programming dual of maximum perfect matching [7, 13]: Each vertex v has a real-valued dual variable $y(v)$. The dual variables *dominate* if for every edge uv,

$$y(u) + y(v) \geq w(uv).$$

Edge uv is *tight* if equality holds; a set of tight edges is *tight*. A set of dual variables is *tight* if there is a tight perfect matching. The *dual objective* is

$$y(V).$$

A perfect matching M is maximum iff it is tight with respect to a set of dominating duals. For the "if" direction we observe that tightness implies $w(M)$ equals the dual objective; dominance implies any perfect matching weighs at most the dual objective. For the opposite direction we now show such duals exist. Note the above discussion shows that dominating tight duals are optimum, i.e., their dual objective equals the weight of a maximum matching.

Without loss of generality assume G has a perfect matching, say M. Let G^+ be G with an additional vertex $s \in V_1$ and weight 0 edges su, $u \in V_0$. For $v \in V_1$, let M_v be a maximum v-matching on G^+. Such a matching exists, for instance if edge $uv \in M$, the v-matching $M - uv + su$. For $u \in V_0$, let the graph G_u be G^+ with another vertex $u' \in V_0$ whose edges are copies of the edges incident to u with the same weights. Let M_u be a maximum perfect matching on G_u. Such a matching exists, for instance $M + su'$.

Theorem 2.1 *A bipartite graph with a perfect matching has dominating tight duals*

$$y(v) = (-1)^i w(M_v)$$

where $v \in V_i$.

Proof: Let uv be an edge with $u \in V_0,\ v \in V_1$. To show dominance, observe that $M_v + u'v$ is a perfect matching on G_u. Hence it weighs no more than M_u, i.e., $w(M_v) + w(uv) \le w(M_u)$, as desired.

Let M be any maximum perfect matching. To show tightness on M it suffices to show that each $uv \in M$ satisfies $w(M_v) \ge w(M_u) - w(uv)$, since we have already shown dominance. This follows if $u'v \in M_u$, since then $M_u - u'v$ is a v-matching on G^+ and so weighs no more than M_v. We will find an M_u containing edge uv; since u and u' are isomorphic this implies G_u has the desired matching.

For any maximum matching M_u, $M \oplus M_u$ contains an augmenting (*wrt* M) su'-path P. Since u and u' are isomorphic we can assume that $u \notin P$ (i.e., if $P = s \ldots tuv \ldots t'u'$, with $tu, t'u' \in M_u$ then modify M_u to contain edges $tu', t'u$ so the path becomes $s \ldots tu'$). M and M_u both induce a maximum perfect matching on $G - V(P)$ so we can assume they are identical on $G - V(P)$. Thus $uv \in M$ implies $uv \in M_u$. $\Box$

We call the duals of the theorem *canonical* because of the following result. Consider any function $y : V \cup s \to \mathbb{R}$ that is dominating and tight on each graph G_v, $v \in V$. This makes sense even for $v \in V_0$ because we can take $y(v')$ to be $y(v)$. Indeed this choice is forced, since if edges vx and $v'x'$ are in a maximum matching, vx' and $v'x$ are also in a maximum matching.

Corollary 2.2 *(i) The dual function y of Theorem 2.1 is dominating and tight on every graph $G_v, v \in V$. So is any translation of y, i.e., for any fixed D,*

$$y'(v) = y(v) - (-1)^i D \textit{ for } v \in V_i. \tag{1}$$

(ii) Any function $y' : V \cup s \to \mathbb{R}$ that is dominating and tight on every graph G_v, $v \in V$, is a translation of y, specifically, $D = y'(V + s)$ in (1).

Proof: (i) Take any graph G_x. An edge uv of G_x with $v \in V_1 - s$ is dominated by y because even if $x \in V_0$ there is a similar edge in G. For an edge us in G_x, recall $M + u's$ is a perfect matching on G_u, so $w(M_s) + w(us) \le w(M_u)$, i.e., dominance holds.

Tightness on G_x is proved by the same argument as the theorem. Alternatively y is tight on G_x if the dual objective is the weight of a maximum matching, $w(M_x)$. We will show this is true, in fact the dual objective on G_x is given by the expression

$$w(M_s) - w(M_s) + w(M_x).$$

In proof, the first term $w(M_s)$ is the weight of a maximum matching on G, so the theorem shows it is the dual objective of y on G. If $x \in V_1$ the dual objective on G_x is obtained from the objective on G by adding the dual of s, $-w(M_s)$, and subtracting the dual of x, $-w(M_x)$. If $x \in V_0$ the dual objective on G_x is obtained from the objective on G by adding the dual of s, $-w(M_s)$, and adding the dual of x', $w(M_{x'}) = w(M_x)$.

The second assertion (1) is clear since translating does not change $y(uv)$ for any edge.

(ii) The hypothesis implies that the weight of a maximum matching on any G_x is given by the sum of the dual variables. For any vertex $x \in V_1$ this means $w(M_x) = y'(V + s - x)$. For $x \in V_0$ it means $w(M_x) = y'(V + s + x')$. It is easy to see these equations are equivalent to the claimed relations. □

3 Bipartite f-factors

Consider a bipartite multigraph $G = (V, E)$ with degree constraint function $f : V \to \mathbb{Z}_0$ and, as before, vertex sets V_0, V_1, and weight function $w : E \to \mathbb{R}$. We are interested in finding a maximum (weight) f-factor. For $x \in \mathbb{R}$, x^+ denotes its positive part, i.e., $x^+ = \max\{x, 0\}$.

We review the linear programming dual problem for maximum f-factors. Each vertex v has a real-valued dual variable $y(v)$. The dual variables *dominate* edge e if $y(e) \ge w(e)$; they *underrate* e if $y(e) \le w(e)$. The duals are *optimal* if some f-factor F consists of underrated edges and its complement consists of dominated edges. The *dual objective* is $fy(V) + (w-y)^+(E)$ (recall $(w-y)^+$ is the positive part of function $w - y$, so $(w - y)(E)$ is the total excess of the underrated edges).

Observe that the above f-factor F for a set of optimal duals is a maximum weight f-factor. In proof, $G - F$ contains no strictly underrated edge. Hence F contains every strictly underrated edge and $w(F)$ equals the dual objective. The dual objective is an obvious upper bound on the weight of any f-factor. So F has maximum weight.

We now show that such optimal duals y always exist. Note this implies that an arbitrary f-factor is maximum iff it consists of underrated edges and its complement $G - F$ consists of dominated edges, both wrt y. Equivalently, its weight is given by the dual objective of optimal duals.

Without loss of generality assume G has an f-factor F. Let G^+ be G with an additional vertex $s \in V_1$ and weight 0 edges su, $u \in V_0$; set $f(s) = 1$. For any vertex v define a degree constraint function f_v to be identical to f on all vertices except v, and $f_v(v) = f(v) + (-1)^i$. (If $v \in V_1$ and $f(v) = 0$, v is irrelevant to an f-factor, so assume it is deleted.) Let F_v be a maximum f_v-factor on G^+. This subgraph exists: for $v \in V_0$ we have $F + sv$; for $v \in V_1$, for any $uv \in F$ we have $F - uv + su$.

Theorem 3.1 *A bipartite multigraph with an f-factor has optimal duals*

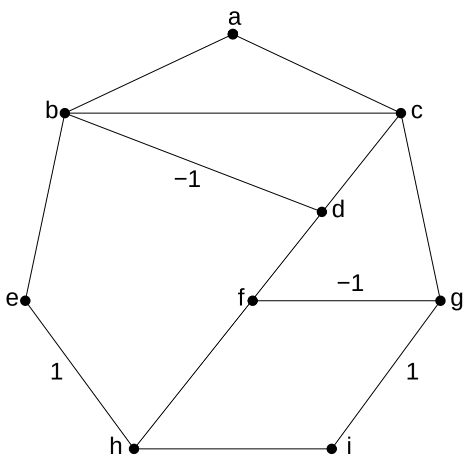
$$y(v) = (-1)^i w(F_v)$$

where $v \in V_i$.

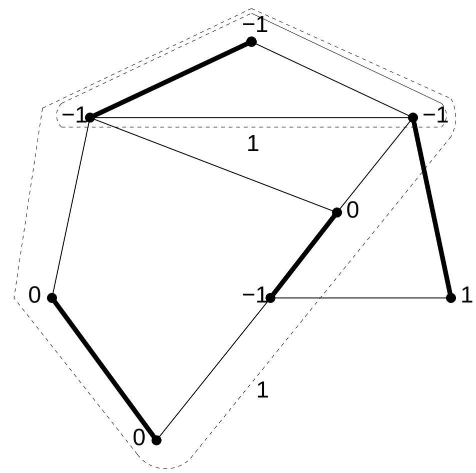
Proof: Let F be a maximum f-factor. Take any edge uv of G, with $u \in V_0, v \in V_1$.

First we show F consists of underrated edges, i.e., $uv \in F$ implies $w(F_v) \ge w(F_u) - w(uv)$. The inequality follows if $uv \in F_u$, since then $F_u - uv$ is an f_v-factor and so weighs no more than F_v.

To show $uv \in F_u$ observe that $F \oplus F_u$ contains an augmenting (*wrt* F) su-path P. By definition $uv \notin P$ (i.e., P ends in an edge $tu \in F_u - F$, so $t \ne v$). Since F and F_u are both maximum we can assume they are identical outside of $E(P)$. So $uv \in F_u$.



(a) Graph and nonzero edge weights.



(b) Maximum i-matching and optimum duals.

Figure 1: Example critical graph for matching.

It remains to show the complement $E-F$ consists of dominated edges. The argument is similar: If $uv \notin F_v$, then $F_v + uv$ is an f_u-factor. Hence $w(F_v) + w(uv) \le w(F_u)$ and uv is dominated. We show $uv \notin F$ implies $uv \notin F_v$ by examining $F \oplus F_v$ as above (i.e., $F \oplus F_v$ contains an even alternating sv-path with last edge $tv \in F - F_v$ so $t \neq u$). $\square$

The analog of Corollary 2.2 holds: First, the f-factor duals are optimal for each function f_x. (To show the duals dominate and underrate for f_x follow the tightness proof of Theorem 2.1, considering $F_x \oplus F_u$ instead of $M \oplus M_u$.) Second, the f-factor duals are canonical. (The argument follows Corollary 2.2(ii) using the dual objective for y', $D = fy'(V) + (w - y')^+(E)$. Any $x \in V_i$ has $w(F_x) = D + (-1)^i y'(x)$, giving (1) as before.)

The theorem also gives an alternate proof of the integral max-flow min-cut theorem. We leave this as an exercise. (The idea is to reduce flow to bipartite f-factor using the equivalence between a directed path $s, a, b, c, d, \ldots, w, x, y, z, t$ and a path $s, a', b, c', d, \ldots, w', x, y', z, t'$ in a bipartite graph with 2 copies v, v' of each network vertex.)

4 Matching on general graphs

This section derives the dual variables for maximum weight perfect matching on general graphs. Fig.1(a) shows a critical graph that will illustrate our discussion. The edge weights are either 0 or ± 1, and edges with nonzero weight are labelled with their weight. Fig.1(b) shows a maximum weight i-matching, where the matched edges are drawn heavy. Vertices and edge sets are labelled with a set of optimum duals, which we now explain.

4.1 Review of matching fundamentals

Let $G = (V, E)$ be an arbitrary graph that has a perfect matching, and let $w : E \to \mathbb{R}$ be a given weight function on G. We use the graph of Fig.1(a) with i deleted as an example, and Fig.1(b) illustrates the following definition. Recall the dual variables from Edmonds' formulation of weighted perfect matching as a linear program [5]: Two functions $y : V \to \mathbb{R}$, $z : 2^V \to \mathbb{R}$, with z

nonnegative except possibly on V, form (a pair of) *dual functions*. (In [5] z is nonzero only on sets of odd cardinality, but it is convenient for us to drop this restriction. Also it is easily reinstated, see below.) Such a pair determines a *dual edge function* $\widehat{yz} : E \to \mathbb{R}$ which for an edge e is defined as

$$\widehat{yz}(e) = y(e) + z\{B : e \subseteq B\}.$$

(Recall that by convention if $e = vw$ then $y(e) = y(v) + y(w)$. Similarly the last term denotes $\sum\{z(B) : e \subseteq B\}$.) The duals *dominate* if for every edge e,

$$\widehat{yz}(e) \geq w(e).$$

The *dual objective* is

$$(y, z)V = y(V) + \sum\{\lfloor |B|/2 \rfloor\, z(B) : B \subseteq V\}.$$

The dual objective upperbounds the weight of any perfect matching M:

$$\begin{aligned} w(M) &= \sum\{w(e) : e \in M\} \\ &\leq \sum\{\widehat{yz}(e) : e \in M\} = \sum\{y(v) : v \in V\} + \sum\{z(B) : e \in M,\ e \subseteq B\} \\ &\leq y(V) + \sum\{\lfloor |V(B)|/2 \rfloor\, z(B) : B \subseteq V\} = (y, z)V \end{aligned} \tag{2}$$

where the last inequality follows since a matching on b vertices has cardinality $\leq \lfloor b/2 \rfloor$.

Edge e is *tight* if the dominance condition holds with equality; a set of tight edges is *tight*. A matching *respects* a set of vertices B if it contains $\lfloor |V(B)|/2 \rfloor$ edges of $\gamma(B)$. A perfect matching is maximum iff it is tight and respects all sets with positive z for a pair of dominating duals. The "if" direction follows from (2). The "only if" direction follows from the existence of such duals. We call such duals y, z *optimum*. Edmonds' blossom algorithm constructively proves the existence of optimum duals [5]. We will give an alternate proof. Note the above discussion shows the weight of a maximum matching equals the objective of optimum duals.

Fig.1(b) shows optimum duals. The optimum duals are not unique, e.g., simpler optimum duals are given by $y(h) = 1$ and all other y, z equal to 0. We will see that the duals of Fig.1(b) come from "canonical" duals.

Before proceeding note that if y, z is a pair of optimum duals, we can modify z to be nonzero only on odd sets: Let $|V(B)|$ be even with $z(B) > 0$ unless $B = V$. Choose an arbitrary vertex $v \in V(B)$, increase $y(v)$ and $z(B - v)$ by $z(B)$ and change $z(B)$ to 0. This transformation does not change $\widehat{yz}(e)$ for $e \subseteq V(B)$. It increases $\widehat{yz}(e)$ for edges going from v to $V - B$, but none of these is matched in M (by tightness of the second inequality of (2)). The dual objective remains unchanged, since $|B|$ even implies $\lfloor |B|/2 \rfloor = \lfloor (|B| - 1)/2 \rfloor + 1$. So it is easy to see that doing this for every such even sets B gives optimum duals with z positive only on odd sets.

A *blossom* is a subgraph B of G defined recursively as follows. For the base case, any vertex is a blossom (it has no edges). For the general case, the vertices $V(B)$ are partitioned into an odd number of sets $V(B_i)$, $1 \leq i \leq k$, $k \geq 3$ odd, where each B_i is a blossom, called a *subblossom* of B. The edges $E(B)$ are the edges $E(B_i)$ plus k more edges that form a cycle on the B_i, i.e., an edge joins $V(B_i)$ to $V(B_{i+1})$, where $k + 1$ is interpreted as 1. Fig.2 gives an example, where the maximal blossom $B1$ has 3 subblossoms and $B2$ has 5 subblossoms. Note that a blossom is not an induced subgraph.

A blossom B can be represented by an ordered tree called a *blossom tree* (see Fig.2). Its root is a node labelled B. If B contains subblossoms B_i, $1 \leq i \leq k$, then the subtrees of the root are the blossom trees for $B_1, \ldots, B_k$ in that order. In addition each node B_i is labelled by the edge $b_i c_i$ of

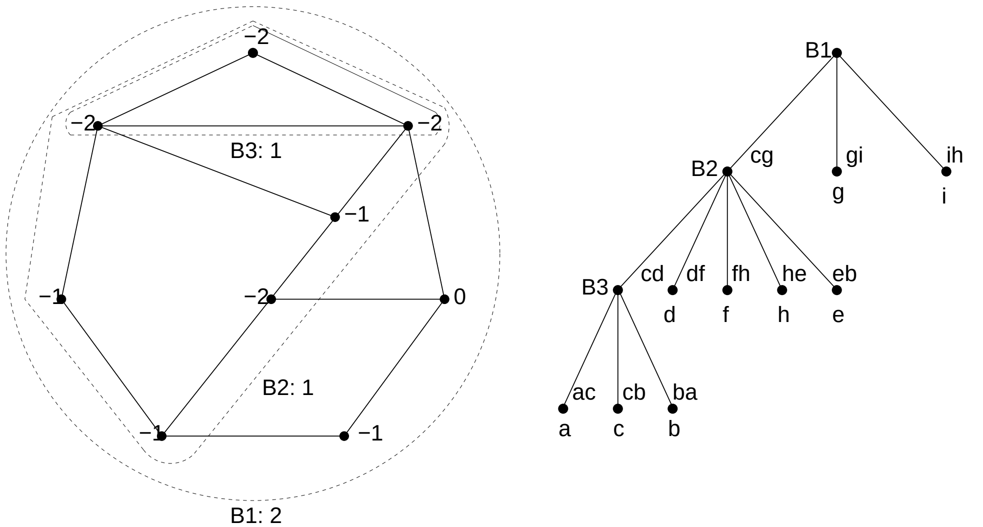


Figure 2: Optimum matching structure for critical graph: Duals y, z and blossom tree.

B that has $b_i \in V(B_i)$, $c_i \in V(B_{i+1})$. A *blossom forest* is a collection of blossom trees such that any vertex is a leaf in at most one tree.

A maximum cardinality matching on a blossom has precisely one free vertex. Such a matching *respects* blossom tree $\mathcal{B}$ if it respects each blossom (i.e., node) of $\mathcal{B}$. In this case each blossom (i.e., node) of $\mathcal{B}$ has a *base vertex*: The base vertex of blossom B is the unique vertex in B not matched to a vertex in B. (Sometimes when context makes it clear, "base" refers to the subblossom B_i containing the base vertex.)

Edmonds' algorithm maintains a *structured matching*. This is a matching M plus blossom forest F plus dual functions y, z that collectively satisfy these conditions:

(i) M respects F.

(ii) z is nonzero only on nonleaf blossoms of F.

(iii) The duals dominate every edge and are tight on every edge that is matched or in a blossom subgraph.

Clearly these conditions imply a structured matching that is perfect is a maximum perfect matching.

Let G be a critical graph. We now define an *optimum matching structure* for G. Its purpose is to provide a succinct representation of a maximum matching M_v for each $v \in V$; our construction will produce this structure. The optimum matching structure consists of a blossom tree $\mathcal{B}$ plus dual functions y, z, such that every vertex is a leaf of $\mathcal{B}$ and properties (ii) and (iii) above are satisfied (see Fig.2). There is no matching in this definition, so property (i) is omitted and (iii) only requires the blossom edges to be tight. It is easy to see that Edmonds' algorithm computes an optimum matching structure when it is executed on a critical graph.

Given an optimum matching structure, for any vertex v a maximum v-matching M_v can be found as follows. We use a recursive procedure $\mathit{blossom\_match}(B, \beta)$, where B is a node of the blossom tree $\mathcal{B}$ and β is the base vertex of B in M_v (e.g., in the initial call B is the root of $\mathcal{B}$ and β is v). Let the subblossoms of B be $B_1, \ldots, B_k$, joined by the k edges $b_j c_j$ as above. Choose i so $\beta \in V(B_i)$. The edges that are matched are $b_j c_j$, $j = i+1, i+3, \ldots, i+k-2$, where as above we assume addition wraps around. Add these to M_v. Now the base β_r of each subblossom B_r is

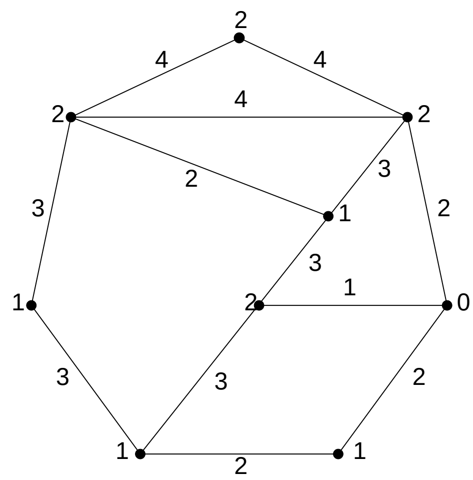


Figure 3: $w(M_v)$ and ζ values; $\zeta^* = 4$.

determined. (It is β for $r = i$ and the matched vertex of B_r for $r \neq i$.) So for each B_r that is not a vertex we execute the recursive call $blossom\_match(B_r, \beta_r)$.

To show *blossom_match* constructs a maximum weight matching, convert the given duals y, z on G to duals on $G - v$ by discarding the value $y(v)$ and setting $z(B - v) = z(B)$ for each B. (Note this can create even sets with positive z values.) It is easy to see that the constructed matching M_v weighs $(y, z)V - y(v)$. (Note that $(y, z)V$ does not change when we redefine z, since $|B|$ odd implies $\lfloor |B|/2 \rfloor = \lfloor (|B| - 1)/2 \rfloor$.) Thus M_v is a maximum v-matching, and we have optimum duals on $G - v$.

Using an appropriate data structure for $\mathcal{B}$ the algorithm finds M_v in $O(n)$ time (e.g., use parent pointers on $\mathcal{B}$, observing that a vertex u of G is the base vertex of blossoms that form a path in $\mathcal{B}$ from u to some ancestor).

4.2 Deriving the duals

We show how to construct an optimum matching structure for a critical graph G, and then we extend the result to perfect matching. The construction, when specialized to the case of unit weight edges, is similar to Lovász's proof that a critical graph has an ear decomposition into odd length ears [8].

In a critical graph, for any vertex v let M_v be a maximum v-matching. For any edge uv define

$$\zeta(uv) = w(M_u) + w(M_v) + w(uv).$$

See Fig.3, where each vertex v is labelled by $w(M_v)$ and each edge uv is labelled by $\zeta(uv)$. Let ζ^* be the maximum value of ζ.

Lemma 4.1 *Any edge e of a critical graph is in an odd cycle C of edges f satisfying $\zeta(f) \geq \zeta(e)$. If $\zeta(e) = \zeta^*$ then any vertex $v \in V$ has a maximum v-matching that respects C.*

Proof: Consider any edge $e = uv$. $M_u \oplus M_v$ contains an alternating (*wrt* M_u or M_v) even uv-path P. Thus $C = P + uv$ is an odd cycle. M_u and M_v both induce a maximum perfect matching on $G - V(C)$. Without loss of generality they induce the same perfect matching, say R.

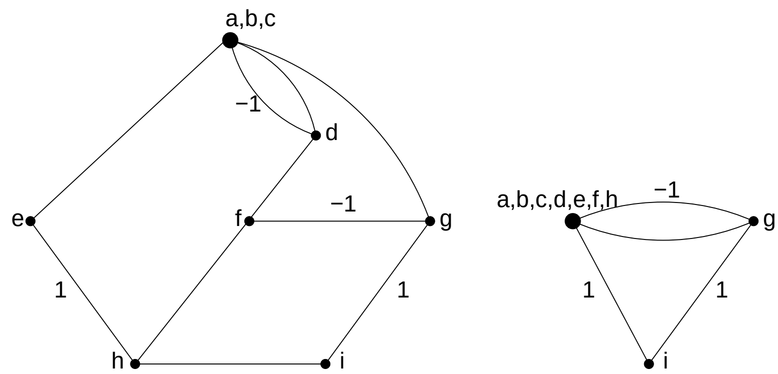


Figure 4: Constructing the blossom tree: Graphs $\overline{G}$ with nonzero edge weights.

Consider any vertex $t \in C$. Define a t-matching N_t to be $R \cup C_t$, where C_t is the (unique) t-matching of C. For any edge rs of C, each edge of $C - rs$ is in precisely one of the matchings C_r, C_s. Thus

$$\zeta(rs) \geq w(N_r) + w(N_s) + w(rs) = w(C) + 2w(R) = w(M_u) + w(M_v) + w(uv) = \zeta(uv),$$

giving the first assertion.

Now assume $\zeta(uv) = \zeta^*$. The above inequality implies $\zeta(rs) = \zeta^*$ and each N_r is a maximum r-matching for $r \in C$. This gives the second assertion of the lemma for vertices r of C.

To complete the proof assume $r \notin C$. Choose any vertex $t \in C$. $M_r \oplus N_t$ contains an alternating even rt-path. Let P be the subpath from r to the first vertex of C, say x. The last edge of P is in M_r, since $N_t \cap \delta(C) = \emptyset$. Since N_x and N_t are identical outside of C, $M_r \oplus N_x$ contains P as a connected component. Thus without loss of generality M_r is identical to N_t on $G - V(P)$. Thus M_r respects C. $\square$

Given an odd cycle C as in Lemma 4.1, define graph $\overline{G}$ to be G with C contracted to a single vertex $\overline{C}$ (see Fig.4). In this section we assume a contraction operation can create parallel edges but not self-loops. So there can be parallel copies of edges from a vertex not in C to $\overline{C}$. Thus an edge e of G with at most one end in C corresponds to a unique edge in $\overline{G}$; for convenience we use e to also refer to the latter edge. Note that the lemma implies $\overline{G}$ is critical (a perfect $\overline{C}$-matching is induced by each N_r, $r \in C$).

For $v \in C$ let C_v be a v-matching on C. For edge uv in $\overline{G}$ define a weight $\overline{w}(uv)$ by

$$\begin{aligned} \overline{w}(uv) &= w(uv) & u, v \notin C, \\ &= w(uv) + w(C_v) & v \in C. \end{aligned}$$

The new weight function preserves weights in the following sense:

(i) For any vertex $v \in V - C$, a maximum v-matching weighs the same in G and $\overline{G}$.
(ii) For any $v \in C$, a maximum $\overline{C}$-matching (in $\overline{G}$) weighs $w(M_v) - w(C_v)$.
(iii) Any edge e of $\overline{G}$ has $\zeta(e)$ the same in G and $\overline{G}$.

To prove (i) use the fact that Lemma 4.1 shows the maximum v-matching in G contains exactly one edge of $\delta(C)$. To prove (ii) use the fact that Lemma 4.1 shows a maximum v-matching (in G) contains C_v and so contains a maximum matching on $G - V(C)$. (iii) follows from (i) and (ii).

Define a blossom tree $\mathcal{B}$ for G by repeating the following step until the graph consists of one vertex: Create a blossom C from an odd cycle given by Lemma 4.1; then change the graph to $\overline{G}$.

The construction can be iterated since as noted $\overline{G}$ is critical. The blossoms so defined give a blossom tree $\mathcal{B}$ that contains every vertex as a leaf. Property (iii) shows ζ^* never increases as the construction progresses.

Now we define dual variables on G using this notation: For a blossom B let $p(B)$ be the parent of B in $\mathcal{B}$, and let $\zeta(B)$ be the value ζ^* when B is created. (For instance $\zeta(V)$ is the last value of ζ^*, which causes the graph to be contracted into a single vertex.) Then define duals

$$\begin{aligned} y(v) &= -w(M_v) & & v \in V, \\ z(B) &= \begin{cases} \zeta(V) & B = V, \\ \zeta(B) - \zeta(p(B)) & B \text{ a blossom} \neq G, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(We get Fig.2.) z is nonnegative function except perhaps on V (since as noted, ζ^* never increases). Clearly any blossom B has $\zeta(B) = z\{C : V(B) \subseteq V(C)\}$.

Theorem 4.2 *For a critical graph G, duals y, z and blossom tree $\mathcal{B}$ form an optimum matching structure.*

Proof: Let e be an edge of G. The definition of y implies that $w(e) = y(e) + \zeta(e)$. Let B be the first blossom created with $e \subseteq V(B)$. Hence $\zeta(e) \le \zeta(B)$. This implies dominance, since

$$w(e) \le y(e) + \zeta(B) = y(e) + z\{C : e \subseteq C\}.$$

If e is an edge of a blossom subgraph the above holds with equality so e is tight. □

Now consider a graph G with a perfect matching M. Let G^+ be G with an additional vertex s and weight 0 edges $sv,\ v \in V$. G^+ is critical: For vertex v with edge $vv' \in M$, $M - vv' + v's$ is a v-matching. So Theorem 4.2 gives an optimum structured matching on G^+. M_s is a maximum perfect matching on G. So we have optimum duals for an s-matching on G^+.

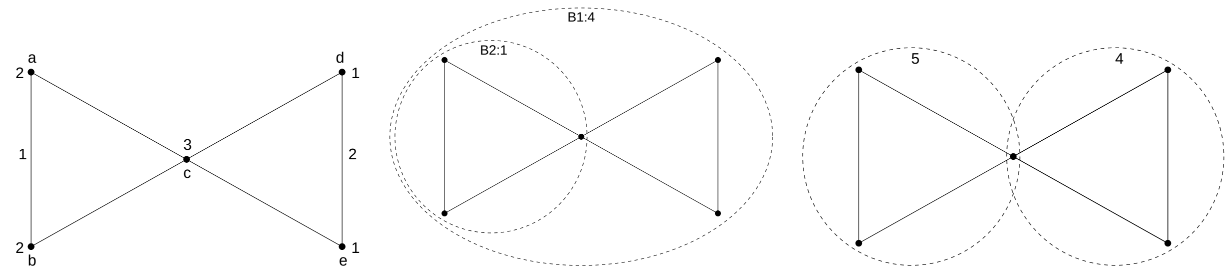
These duals are essentially optimum duals for a maximum perfect matching on G. It is easy to modify these duals to get the standard optimum duals (as in [5]): For every B containing s, replace B by $B - s$; then use the transformation of Section 4.1 to eliminate this even set from the support of z.

We next argue that our duals are canonical. Define *optimum duals for a critical graph G* to be a pair of functions y, z that is dominating, tight on every maximum matching M_v, each such M_v respects every set with positive z, and z is nonzero only on odd sets. This implies every maximum M_v has weight equal to the dual objective $(y, z)V - y(v)$. (Note that $(y, z)V$ has the same value on G and $G - v$, since odd sets B containing v have $\lfloor |B|/2 \rfloor = \lfloor (|B| - 1)/2 \rfloor$.)

It is convenient to switch notation and denote our duals as y^*, z^*. We are interested in how y^*, z^* relate to arbitrary optimal duals y, z on the critical graph G. Obviously our duals give other optimum duals by translation– for any value D, decrease each $y^*(v),\ v \in V$, by D and increase $z^*(G)$ by $2D$. Conversely any optimum y is a translation of y^*. In proof for any $v \in V$, $(y, z)(V - v) = w(M_v)$, so $y(v) = (y, z)V - w(M_v) = (y, z)V + y^*(v)$.

In light of this relation, for the rest of the discussion translate y, z so $z(G) = z^*(G)$. The functions z and z^* obey an obvious relation: For any edge e define

$$\bar{z}(e) = z\{X : e \subseteq X\}$$



(a) Graph, nonzero edge weights and matching weights. (b) Blossoms and z values. (c) z with nonlaminar support.

Figure 5: z needn't have laminar support.

and define $\bar{z}^*(e)$ analogously for the function z^*.[3] Let EM be the set of edges in some maximum matching M_v, $EM = \bigcup_v M_v$. Tightness means

$$e \in EM \Longrightarrow \bar{z}^*(e) = \bar{z}(e).$$

However z and z^* needn't have the same support: Fig.5 gives an optimum z with nonlaminar support.

Requiring laminarity makes our duals *canonical*, as the next result shows. We continue to assume that y, z is a pair forming optimum duals for the critical graph G and we have normalized to make $z(G) = z^*(G)$.

Theorem 4.3 *If the support of z is laminar then $z = z^*$.*

Proof: Let $\mathcal{Z}$ ($\mathcal{Z}^*$) denote the support of z (z^*), respectively. Clearly it suffices to show these supports are the same. Consider an arbitrary set $Z \in \mathcal{Z}$.

Claim 1 *If Z contains edge e and is crossed by edge f then $\bar{z}(e) > \bar{z}(f)$.*

Take $X \in \mathcal{Z}$ with $f \subseteq X$. The latter implies $X \cap Z \neq \emptyset$ and $X \not\subseteq Z$. Thus $Z \subset X$. So the sets of $\mathcal{Z}$ contributing to $\bar{z}(f)$ are a proper subset of those contributing to $\bar{z}(e)$, $\bar{z}(e) > \bar{z}(f)$. $\diamondsuit$

Claim 2 *Z is connected by the edges EM.*

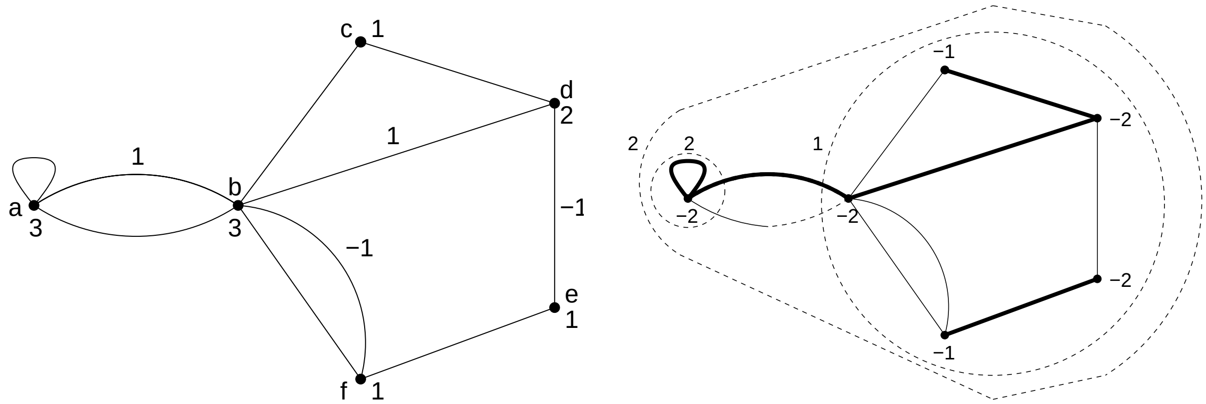
Suppose Z is the union of disjoint sets Z_0, Z_1 with no edge of EM joining them. Take $v \in Z$ and consider matching M_v. If $v \in Z_0$ then $|Z_1|$ is even, since M_v respects Z. Similarly $|Z_0|$ is even. This makes $|Z|$ even, but Z is an odd set. $\diamondsuit$

Observe that EM is identical to the set $E(R)$ for R the root of $\mathcal{B}$ (recall the *blossom_match* procedure).

Claim 3 *Consider any $B \in \mathcal{Z}^*$ and any edge $e \in EM$ that is contained in B but no smaller blossom of $\mathcal{Z}^*$. Any set $Z \in \mathcal{Z}$ that contains e actually contains B, $V(E(B)) \subseteq Z$.*

Recall that $E(B)$ makes B a connected subgraph (for any blossom). So arguing by contradiction, let $f \in E(B)$ cross Z. Claim 1 shows $\bar{z}(e) > \bar{z}(f)$. But every edge $g \in E(B)$ has

[3] $\bar{z}^*(e)$ is essentially $\zeta(e)$ but we don't use this.



(a) Graph with degree constraints and nonzero edge weights.

(b) Maximum f_b-factor and optimum duals.

Figure 6: Example critical graph for f-factors.

$\bar{z}(g) = \bar{z}^*(g) \geq \bar{z}^*(e) = \bar{z}(e)$, contradiction. ♢

Claim 4 $\mathcal{Z}^* \subseteq \mathcal{Z}$.

Take any $B \in \mathcal{Z}^*$. Take an edge $e \in EM$ as in Claim 3 for B. Let $Z \in \mathcal{Z}$ be the minimal set of $\mathcal{Z}$ that contains e. We will show $Z = B$. Claim 3 shows $B \subseteq Z$. So assume the containment is proper, say $v \in Z - B$.

M_v respects Z, so no edge of M_v crosses Z. M_v contains an edge $f \in \delta(B)$, since B is an odd set. Thus $f \subseteq Z$. This implies $\bar{z}(f) \geq \bar{z}(e)$. This is equivalent to $\bar{z}^*(f) \geq \bar{z}^*(e)$ since both edges are in EM.

But $f \in \delta(B)$ and $z^*(B) > 0$ gives $\bar{z}^*(f) < \bar{z}^*(e)$, contradiction. ♢

Claim 4 shows we can complete the proof by showing $\mathcal{Z} \subseteq \mathcal{Z}^*$. So take any $Z \in \mathcal{Z}$. Let B be the minimal blossom in $\mathcal{Z}^*$ that contains Z. It suffices to show $Z = B$. In fact we need only show $B \subseteq Z$.

The minimality of B shows Z contains two vertices not both in the same set B' for any $B' \in \mathcal{Z}^*$ and $B' \subset B$. Claim 2 shows Z is connected in EM. Thus Z contains an edge $e \in EM$ that is contained in B but no smaller $B' \in \mathcal{Z}^*$. Claim 3 shows $B \subseteq Z$. □

5 *f*-factors on multigraphs

This section derives the dual variables for the maximum weight f-factor problem on arbitrary multigraphs. As before, Fig. 6(a) shows a graph that will illustrate our discussion. Edge weights are in $\{0, \pm 1\}$, edges with nonzero weight are labelled with their weight, and vertices are labelled with their degree constraint $f(v)$. Fig. 6(b) shows a maximum weight f_b-factor, where f_b has $f_b(b) = 2$ and is otherwise identical to f (see below); the edges in the f_b-factor are drawn heavy. Vertices and edge sets are labelled with a set of optimum duals, which we now explain.

5.1 Review of fundamentals

We review the linear programming dual problem for maximum weight f-factors [13, Ch.32].

Let $\mathcal{I} \subseteq 2^V \times 2^E$ be the family of all pairs (C, I) where $C \neq \emptyset$ and I is a (possibly empty) subset of $\delta(C)$. For $B \in \mathcal{I}$ we write $C(B)$ and $I(B)$. (Sometimes when context allows we write $I(C)$ for the second component of a pair with first component C.) A pair $(C, I) \in \mathcal{I}$ *covers* every edge of $\gamma(C) \cup I$. The main property of a pair $(C, I) \in \mathcal{I}$ is

Any f-factor F contains $\leq \lfloor \frac{f(C)+|I|}{2} \rfloor$ edges covered by (C, I).

This follows since a covered edge of F contributes 2 to the quantity $f(C) + |F \cap I|$. So F contains $\leq f(C) + |I|$ covered edges. An f-factor F *respects* (C, I) if it contains exactly $\lfloor \frac{f(C)+|I|}{2} \rfloor$ covered edges. An *f-odd pair* is an ordered pair $(C, I) \in \mathcal{I}$ with $f(C) + |I|$ odd. We write *odd pair* when f is understood. Similarly for *f-even pair*.

Fig. 6(b) uses the three pairs $(\{a\}, \{ab_1, ab_0\})$, $(V - a, \{ab_1\})$ and $(V, \emptyset)$, where the two parallel edges ab are differentiated by using their weight as a subscript. ab_0 is drawn solid on the a side and dashed on the b side to show that it belongs to $I(a)$ but not $I(V - a)$. The f_b-factor respects all these pairs, since $\lfloor 5/2 \rfloor = 2$ and $\lfloor 8/2 \rfloor = 4$.

Two functions $y : V \to \mathbb{R}$, $z : \mathcal{I} \to \mathbb{R}$ form (a pair of) *dual functions* if $z(C, I) \geq 0$ whenever $C \subset V$. Such a pair determines a *dual edge function* $\widehat{yz} : E \to \mathbb{R}$, defined by

$$\widehat{yz}(e) = y(e) + z\{(C, I) : (C, I) \in \mathcal{I} \text{ covers } e\}.$$

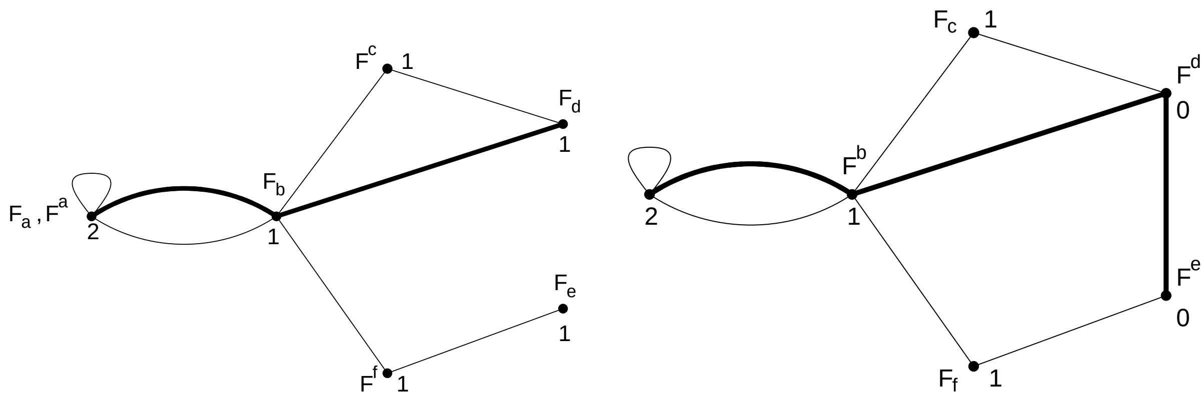
The duals *dominate* edge e if $\widehat{yz}(e) \geq w(e)$; they *underrate* e if $\widehat{yz}(e) \leq w(e)$. (In Fig. 6(b) edge bf_{-1} is strictly dominated, bd is strictly underrated, and all other edges are tight.) As in Section 3, $(w - \widehat{yz})^+(E)$ is the total excess. Any f-factor F satisfies

$$\begin{aligned} w(F) &\leq \sum\{\widehat{yz}(e) : e \in F\} + (w - \widehat{yz})^+(E) \\ &= fy(V) + \sum\{z(C, I) : e \in F,\ e \text{ covered by } (C, I)\} + (w - \widehat{yz})^+(E) \\ &\leq fy(V) + \sum\{\lfloor \frac{f(C) + |I|}{2} \rfloor z(C, I) : (C, I) \in \mathcal{I}\} + (w - \widehat{yz})^+(E) \\ &= (y, z)V \end{aligned} \tag{3}$$

where the equality in the last line defines the *dual objective* $(y, z)V$. F is maximum iff its weight equals the value of the dual objective for some pair y, z, i.e., F consists of underrated edges, its complement consists of dominated edges, and it respects all pairs with positive z. ((In Fig. 6(b) the f_b-factor weighs 2 and the dual objective is $-2(3+2+2+1) - 1(1+1) + 2\lfloor 10/2 \rfloor + 2\lfloor 5/2 \rfloor + 1\lfloor 8/2 \rfloor + 2 = -16 - 2 + 10 + 4 + 4 + 2 = 2$.) The "if" direction follows from (3). The "only if" direction follows from the existence of such duals [13, Ch.32]. We call such duals y, z *optimal*, so the weight of a maximum f-factor equals the objective of optimal duals. We give a combinatorial proof that optimal duals always exist.

For each vertex $v \in V$ define f_v, the *lower perturbation* of f at v. by decreasing $f(v)$ by one. Similarly define f^v, the *upper perturbation* of f at v. by increasing $f(v)$ by one. Each f_v, f^v, $v \in V$ is a *perturbation* of f. The notation $f\updownarrow_v$ stands for a fixed perturbation that is either f_v or f^v. A graph is *f-critical* if it has an f'-factor for every perturbation f' of f.

The graph of Fig. 6(a) is critical. This can be seen by examining Fig. 7, which gives a maximum weight factor $F\updownarrow_v$ for every perturbation $f\updownarrow_v$. The perturbations which have a weight 2 factor are



(a) Maximum factors of weight 2 with residual degree constraints.

(b) Maximum factors of weight 1 with residual degree constraints.

Figure 7: Maximum perturbed f-factors.

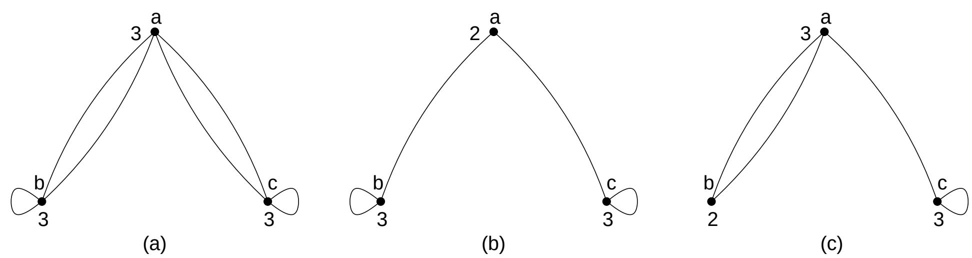


Figure 8: Noncritical graph: (a) Graph and degree constraints. (b) f_a-perturbation. (c) f_b-perturbation. No f^a-perturbation exists.

indicated in Fig. 7(a) – vertex labels specify the factors with weight 2, each such factor contains the heavy edges plus additional edges selected according to the residual degree constraints shown (e.g., the additional edges in F_b are shown in Fig. 6(b)). The remaining perturbations have a maximum factor of weight 1 and are indicated in Fig. 7(b).

The definition of criticality is consistent with matching criticality. Indeed for an arbitrary degree constraint f, if an f_u-factor exists and omits some edge uv incident to u, adding it gives an f^v-factor. In a matching-critical graph both conditions always hold, i.e., when all lower perturbations exist so do all upper perturbations. Fig.8 shows this is not true for arbitrary f. (We have chosen a bridgeless noncritical graph since Lemma 5.10 below shows bridgeless graphs have special properties.)

For an f-critical graph the dual pair y, z is *optimal* if z is nonzero only on f-odd pairs and every maximum perturbation of f satisfies the above optimality condition, i.e., a maximum f_v-factor weighs

$$\begin{aligned} &(y,z)V - y(v) \\ &= \ fy(V) - y(v) + \sum\{\lfloor \tfrac{f(C)+|I|}{2} \rfloor z(C,I) : (C,I) \in \mathcal{I}\} + (w - \widehat{yz})^+(E) \end{aligned} \tag{4}$$

and a maximum f^v-factor weighs

$$(y,z)V + y(v) + \sum\{z(C,I) : (C,I) \in \mathcal{I},\ v \in C\}. \tag{4'}$$

(We use f-oddness.) We will show how to construct optimal duals for an f-critical graph G (and then extend that to f-factors, as in matching). In fact we will construct an "optimum f-factor structure" analogous with matching, but that concept won't be needed for our development.

The next lemma characterizes when an f-factor respects a pair.

Lemma 5.1 *An f-factor F respects $B=(C,I)\in\mathcal{I}$ if and only if*

$$F\cap\delta(C)=\begin{cases} I & B \text{ is } f\text{-even} \\ I\oplus e & B \text{ is } f\text{-odd, } e \text{ is some edge in } \delta(C). \end{cases}$$

Proof: Let δ be 0 (1) if B is f-even (f-odd). F respects B iff $\lfloor\frac{f(C)+|I|}{2}\rfloor = d(F,\gamma(C))/2+d(F,I)$, i.e.,

$$f(C)+|I|-\delta = d(F,\gamma(C))+2d(F,I)=f(C)+d(F,I)-d(F,\delta(C)-I).$$

Clearly $d(F,I)\le |I|$. So comparing the left-hand side with the right shows when $\delta=0$ the condition is equivalent to $d(F,I)=|I|$, $d(F,\delta(C)-I)=0$. Similarly when $\delta=1$ the condition is equivalent to $d(F,I)=|I|-1$, $d(F,\delta(C)-I)=0$ or $d(F,I)=|I|$, $d(F,\delta(C)-I)=1$. The lemma follows. $\Box$

The proof does not refer to values of f outside of C. So in our context, the lemma shows that an $f\!\updownarrow\! v$-factor F respects an f-odd pair $B=(C,I)$ iff $F\cap\delta(C)$ is I when $v\in C$, or $I\oplus e$ for some $e\in\delta(C)$ when $v\notin C$.

Note the graph of Fig.8 does not have the analog of optimum f-critical duals, i.e., every lower perturbation exists but no dual pair y,z is optimum for every lower perturbation. In proof, assign weight 1 to each loop and 0 to each nonloop. (4) shows that wlog we can assume $y(a)=-2$, $y(b)=y(c)=-1$, The perturbations of Fig.8(b)–(c) show every edge e is in some maximum perturbation and not in another. Thus e is tight. So each loop (nonloop) is covered by sets of total z-value 2 (3). Thus some pair (C,I) with positive z covers ab but not bb. Clearly $C=\{a\}$. Hence every maximum f_a-perturbation F has $F\cap\delta(a)$ equal to I or $I\oplus e$, i.e., two such perturbations differ on ≤ 2 edges of $\delta(a)$. Fig.8(a) shows there are two such perturbations differing on all 4 edges of $\delta(a)$.

5.2 Unifactors

This section presents the properties of "unifactors", the subgraphs that we use to find blossoms.

As in matching, blossoms are built up from odd cycles. The "odd cycle" part is captured in the following definition.

Definition 5.2 *An* elementary blossom *B is a 4-tuple $(VB, C(B), CH(B), I(B))$, where $VB\subseteq V$, $C(B)$ is an odd circuit on VB, $CH(B)\subseteq\gamma(VB)$, $I(B)\subseteq\delta(VB)$, and every $v\in VB$ has*

$$f(v)=d(v,C(B))/2+d(v,CH(B)\cup I(B)). \tag{5}$$

We call $C(B), CH(B)$, and $I(B)$ the *circuit, chords*, and *incident edges* of B, respectively. Note that we allow VB to be a singleton $\{v\}$, in which case $C(B)$ is an odd number of loops vv, $CH(B)$ contains other loops vv, and $I(B)$ contains edges incident to B.

In general $(VB, I(B))$ forms an odd pair, since summing the equations (5) gives

$$f(VB)+|I(B)|=|C(B)|+2|CH(B)|+2|I(B)|\equiv 1 \pmod 2.$$

From now on all congruences will be modulo 2 and we will omit the modulus. Also for convenience we sometimes use the term "blossom" or "elementary blossom" to reference the blossom's circuit or its odd pair.

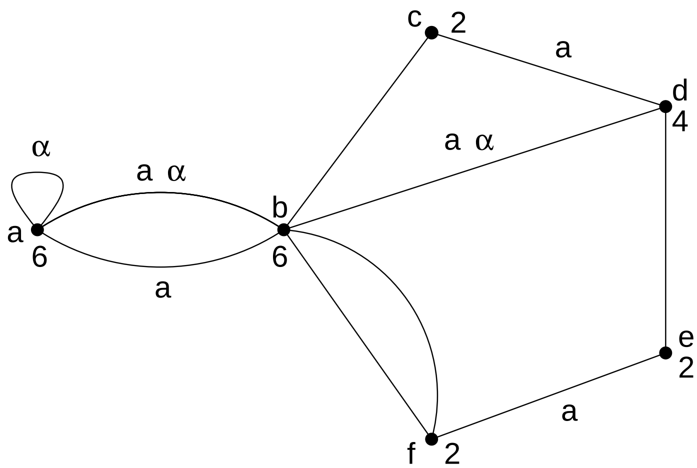


Figure 9: Multiplicity 2 edges of unifactors for circuits on a and $\alpha = V - a$. Vertex labels give $2f$.

For simplicity perturb the edge weights slightly so that no two sets of edges have the same weight. That is, number the edges from 1 to m and increase the weight of the ith edge by ϵ^i for some $\epsilon \geq 0$. For small enough $\epsilon > 0$, no two sets of edges have the same weight. Thus any such perturbation has a unique maximum factor which is also maximum for the original weights. Let F_v and F^v denote the maximum weight f_v and f^v -factors respectively.

We use this terminology for a multiset S. $2S$ denotes S with every multipicity doubled. Similarly for a multigraph $G = (V, E)$, $2G$ denotes the multigraph $(V, 2E)$. If every multiplicity of S is even then $S/2$ denotes S with every multiplicity halved.

A $2f$*-unifactor* is a $2f$-factor of $2G$ whose edges of odd multiplicity form an odd circuit. Equivalently it consists of edges of multiplicity two plus an odd circuit of multiplicity one. (Parallel edges of multiplicity two allow us get multiplicities > 3.) In the case of matching (i.e., $f \equiv 1$) a $2f$-unifactor is a 2-matching with one cycle [10]. Note also that a $2f$-unifactor can be viewed as "almost bipartite". We often abbreviate "$2f$-unifactor" to "unifactor" without causing confusion.

In Fig.9 and later figures it is convenient to define the edge set

$$\alpha = V - a.$$

The figure shows the multiplicity two edges in the unifactors whose circuits span a and α respectively. These two unifactors weigh 4 and 3 respectively and are the two largest unifactors.

We shall use the maximum weight unifactor to find the first blossom (and similar unifactors thereafter). Let U be an arbitrary $2f$-unifactor with circuit C. Define the *elementary blossom of* U as the 4-tuple $B = (V(C), C, CH(B), I(B))$ where $CH(B) = (U \cap \gamma(C))/2$ and $I(B) = (U \cap \delta(C))/2$. (As an example, the unifactor for a in Fig.9 corresponds to the elementary blossom for the odd pair $(\{a\}, \{ab_1, ab_0\})$ in Fig. 6(b).)

Lemma 5.3 *B is an elementary blossom. If U is the maximum weight unifactor then C is a cycle.*

Proof: For the first assertion we must check $f(v)$, $v \in C$. The definition of unifactor shows $2f(v) = d(v, C \cup CH(B) \cup I(B))$ (since in a multigraph the degree function d counts edges according to their multiplicity). Dividing by 2 gives (5).

We prove the second assertion by contradiction. Suppose a vertex v has degree > 2 in C. We can write C as the disjoint union of circuits C_1 and C_2 that are joined at v. Let C_1 be odd and

C_2 be even. C_2 is the disjoint union of edge sets D_1 and D_2 formed by taking alternate edges of C_2. Wlog $w(D_1) > w(D_2)$. It is easy to see that $U + D_1 - D_2$ is a $2f$-unifactor with circuit C_1 and weight greater than $w(U)$. But U has maximum weight, contradiction. $\square$

It is easy to see that for a given vertex v, and the maximum unifactor containing v in its circuit C, C need not be a cycle. However the above argument shows that C contains exactly 2 edges (or 1 loop) incident to v.

In the following discussion recall that G is a multigraph that may contain loops. The next lemma and its corollary identify the maximum weight unifactor satisfying certain conditions – the existence of such unifactors may not be immediately obvious but existence is proved as well.

Lemma 5.4 *For any vertex v, the maximum weight $2f$-unifactor containing v in its circuit is $F_v + F^v$.*

Proof: Clearly $F_v + F^v$ is a $2f$-factor. Write $F_v + F^v = 2(F_v \cap F^v) + (F_v \oplus F^v)$, where the edges in the first set have multiplicity two and those of the second set have multiplicity one. The second set $F_v \oplus F^v$ is an odd circuit containing v. Thus $F_v + F^v$ is a $2f$-unifactor.

Let U be a $2f$-unifactor whose circuit C contains v. Halve the multiplicity of each even-multiplicity edge of U, and choose alternate edges of C, starting and ending with two edges at v. We get an f^v-factor U^v. Similarly we get an f_v-factor U_v by omitting the first and last edges at v. Thus

$$w(U) = w(U_v) + w(U^v) \leq w(F_v) + w(F^v) \leq w(U). \tag{6}$$

We conclude that equality holds throughout, proving the lemma. $\square$

It is worth reviewing the proof for the case $vv \subset C$. There are two choices for U_v, depending on the direction that C is traversed. Similarly for U^v. This seems to contradict the uniqueness of F_v and F^v. But this can never occur: If the two traversals yield the partition of $C - vv$ into sets C_1 and C_2, where wlog $w(C_1) > w(C_2)$, then $U + C_1 - C_2$ is a $2f$-unifactor containing v in its circuit and weighing more than U, impossible.

Next we give a version of the lemma that is oriented towards edges rather than vertices v. For ordinary matching, every edge satisfies part (i). Let uv be an arbitrary edge. For a multigraph this means a fixed copy of uv. We allow uv to be a loop, although this makes part (ii) below vacuous. Fig.10 illustrates the corollary. For instance the maximum unifactor containing edge bd in its circuit consists of the length 3 circuit $bdcb$ plus multiplicity two edges de, bf_0, ab_1, aa (Fig.7(b)). As indicated in Fig.10 it weighs 1.

Corollary 5.5 *(i) If $uv \notin F_u \cup F_v$ the maximum weight $2f$-unifactor containing uv in its circuit is $F_u + F_v + uv$.*

(ii) If $uv \in F_v - F_u$ the maximum weight $2f$-unifactor whose circuit contains v and is incident to uv is $F_u + F_v + uv$. Furthermore $F_u = F^v - uv$.

(iii) If $uv \in F_u \cap F_v$ the maximum weight $2f$-unifactor containing uv in its circuit is $F^u + F^v - uv$.

Proof: (i) Suppose $uv \notin F_u \cup F_v$. $F_u + F_v + uv$ is clearly a subgraph of $2G$. If uv is not a loop then

$$d(u, F_u + F_v + uv) = (f(u) - 1) + f(u) + 1 = 2f(u)$$

and similarly for v. A similar equation holds if uv is a loop. We conclude $F_u + F_v + uv$ is a $2f$-factor. $F_u \oplus F_v + uv$ is an odd circuit of edges of multiplicity one, containing uv. So $F_u + F_v + uv$ is a $2f$-unifactor containing uv in its circuit. The analog of the second inequality of (6) holds.

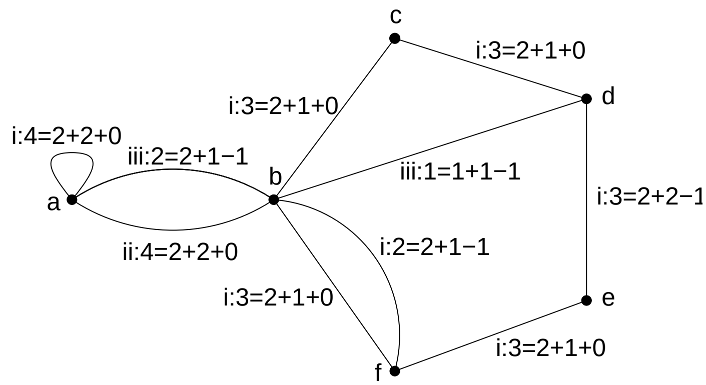


Figure 10: Weight of unifactors from Corollary 5.5: Part number (i) – (iii) and arithmetic expression for the weight.

For the converse let U be the unifactor of (i). The multiplicity one edges of $U - uv$ form an even trail from u to v. Construct U_u and U_v from $U - uv$ by taking alternate edges of the trail, with the first edge in U_v and the last edge in U_u. The analog of the first inequality of (6) holds. Part (i) follows.

(ii) Suppose $uv \in F_v - F_u$. $F_u + F_v + uv$ is a subgraph of $2G$. As in (i) it is a $2f$-factor. $F_u \oplus F_v$ is an even trail from u to v. It starts with the chosen copy of edge uv in $F_v - F_u$, and proceeds along a circuit C at v. So the edges of multiplicity one in $F_u + F_v + uv$ are those of C. In other words $F_u + F_v + uv$ is a $2f$-unifactor whose circuit contains v and is incident to uv. The analog of the second inequality of (6) holds.

Conversely take U as the unifactor of (ii). We show $U - uv$ is the disjoint union of U_u and U_v, respectively f_u and f_v -factors, with $uv \in U_v - U_u$. This will give the analog of the first inequality of (6), proving part the first claim of (ii).

The multiplicity one edges of $U - uv$ form an even trail starting with uv and followed by a circuit containing v. Construct U_u and U_v by taking alternate edges of the trail, with the first edge uv in U_v and the last edge in U_u. Also split a copy of each even multiplicity edge of $U - uv$ into each of U_u, U_v. It is easy to see u and v have degree $f(u)$ and $f(v) - 1$ in U_v, and $f(u) - 1$ and $f(v)$ in U_u, as desired.

We turn to the second claim of (ii). The hypothesis $uv \in F_v - F_u$ shows that adding uv to F_u gives an f^v-perturbation. Hence $w(F_u) + w(uv) \leq w(F^v)$. We have $F_v \cap \delta(v) \subseteq F^v$ (see Lemma 5.4). So $uv \in F^v$, $F^v - uv$ is an f_u-perturbation and so $w(F^v) - w(uv) \leq w(F_u)$. Together the two inequalities imply equality. The perturbed edge weights imply $F^v - uv = F_u$.

(iii) Suppose $uv \in F_u \cap F_v$. This implies $uv \in F^u \cap F^v$. We will show that for any edge uv belonging to $F^u \cap F^v$, the maximum weight $2f$-unifactor containing uv in its circuit is $F^u + F^v - uv$. This implies part (iii).

Suppose $uv \in F^u \cap F^v$. Clearly $F^u + F^v - uv$ is a $2f$-factor. $F^u \oplus F^v$ is an even trail from u to v. So the edges of multiplicity one in $F^u + F^v - uv$ form a circuit containing uv. The analog of the second inequality of (6) holds.

Conversely take U the unifactor of (iii). We will partition the edges of $U + uv$ into subgraphs U^u and U^v, where U^u and U^v are f^u and f^v -factors respectively, both containing uv. This will give the analog of the first inequality of (6), proving part (iii).

The multiplicity one edges of $U - uv$ form an even trail from u to v. As in part (ii) construct U_u and U_v by taking alternate edges of the trail, with the first edge in U_v and the last edge in U_u. Also split a copy of each even multiplicity edge of $U - uv$ into each of U_u, U_v. As in part (ii) u and v have degree $f(u)$ and $f(v) - 1$ in U_v, and $f(u) - 1$ and $f(v)$ in U_u. The subgraph $U^u = U_v + uv$ is an f^u-factor, and $U^v = U_u + uv$ is an f^v-factor. Clearly $U + uv$ can be partitioned into into U^u and U^v. □

The next lemma shows how the maximum weight unifactor gives the first blossom (as in Section 4). As with the preceding lemma and corollary we will give a more involved argument for the general case. Call any subgraph F_v, F^v, $v \in V$ a *maximum perturbation* of f. For brevity the next lemma and its proof refer to "elementary blossom" when we actually mean the f-odd pair of the elementary blossom.

Lemma 5.6 *Every maximum perturbation $F{\updownarrow}v$ respects the elementary blossom of a maximum $2f$-unifactor.*

Proof: Let U be a maximum weight $2f$-unifactor. Let C be the circuit of U, $I = (\delta(C) \cap U)/2$, and $B = (C, I)$ the elementary blossom of U.

Take any vertex $v \in C$. The proof of Lemma 5.4 shows F_v and F^v both consist of alternate edges of C plus the edges $(U - C)/2$. In particular we have this property:

$$(*) \qquad F{\updownarrow}v - \gamma(C) = (U - \gamma(C))/2.$$

From now on the only property of a maximum perturbation $F{\updownarrow}v$ that we use, aside from the definition, is (*). This will allow us to claim the proof holds in the general setting below.

(*) shows $F{\updownarrow}v \cap \delta(C) = I$. So F_v and F^v both respect B.

To complete the proof take any $u \notin C$ and consider a perturbation $f{\updownarrow}u$ and maximum $f{\updownarrow}u$-factor $F{\updownarrow}u$. Taking any $v \in C$, $F{\updownarrow}u \oplus F_v$ is an alternating uv-trail. Let T be the subtrail from u to the first vertex in C, say x. Let e (g) be the first (last) edge of T, respectively. (Note we may have $e = g$. Also T may contain edges incident to I.)

Observe

$$f{\updownarrow}u = \begin{cases} f_u & e \in F_v - F{\updownarrow}u, \\ f^u & e \in F{\updownarrow}u - F_v. \end{cases}$$

Choose the perturbation

$$f{\updownarrow}x = \begin{cases} f_x & g \in F{\updownarrow}u - F_v, \\ f^x & g \in F_v - F{\updownarrow}u. \end{cases}$$

Let $F{\updownarrow}x$ be the corresponding maximum $f{\updownarrow}x$-factor. (*) implies $F{\updownarrow}x$ is identical to F_v on edges not in $\gamma(C)$. So the edges of T alternate between and $F{\updownarrow}u$ and $F{\updownarrow}x$.

In $F{\updownarrow}u \oplus T$, the two displayed equations show u has degree $f(u)$ and x has degree $f{\updownarrow}x(x)$. Thus $F{\updownarrow}u \oplus T$ is an $f{\updownarrow}x$-factor. So it weighs no more than $F{\updownarrow}x$, i.e.,

$$w(F{\updownarrow}u \oplus T) = w(F{\updownarrow}u) + w(T \cap F{\updownarrow}x) - w(T \cap F{\updownarrow}u) \le w(F{\updownarrow}x).$$

In $F{\updownarrow}x \oplus T$, the two displayed equations show u has degree $f{\updownarrow}u(u)$ and x has degree $f(x)$. Thus $F{\updownarrow}x \oplus T$ is an $f{\updownarrow}u$-factor and it weighs no more than $F{\updownarrow}u$:

$$w(F{\updownarrow}x \oplus T) = w(F{\updownarrow}x) - w(T \cap F{\updownarrow}x) + w(T \cap F{\updownarrow}u) \le w(F{\updownarrow}u).$$

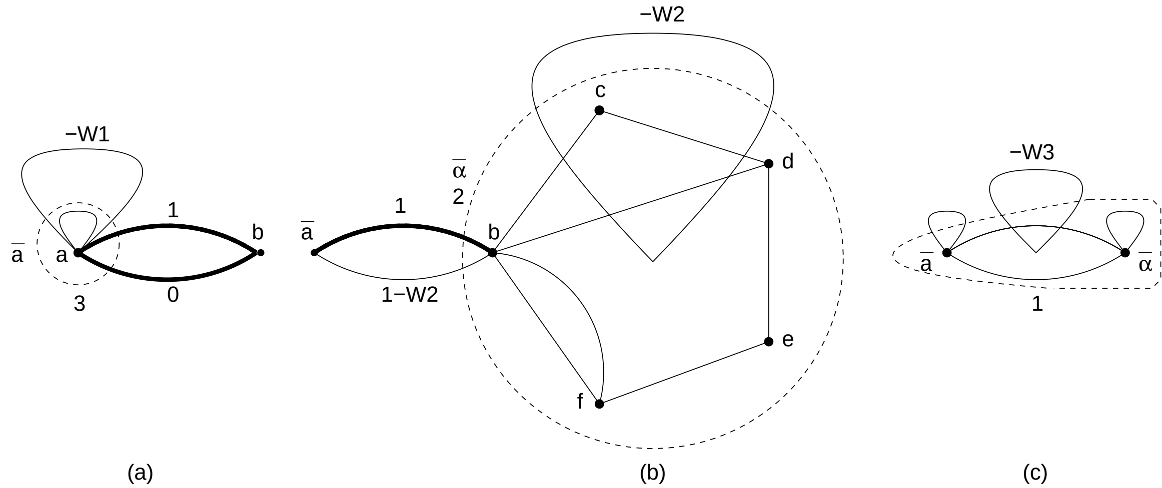


Figure 11: Formation of contracted graphs for Fig.6: new weights and new degree constraints. Heavy edges are in the blossom's I set.

Combining the two preceding displayed inequalities shows equality holds throughout. This implies $F{\updownarrow}u = F{\updownarrow}x \oplus T$. It respects B since $F{\updownarrow}u \cap \delta(C) = I \oplus g$. $\square$

We shall iterate this construction as in Section 4, contracting each blossom B as it is found. Fig.11 shows the three contractions for our example graph. The edge weights in this figure will be explained in Section 5.3, but we note that $W_1 \ll W_2 \ll W_3$. As before $\alpha = V - a$.

We shall see that contracting B forms a new vertex $\overline{B}$ with a loop $\overline{B}\,\overline{B}$ representing B. There will be a corresponding unifactor U whose circuit is $\overline{B}\,\overline{B}$. Such a unifactor represents the already-discovered B and is thus redundant. A unifactor whose circuit is *not* a loop $\overline{B}\,\overline{B}$ that was introduced by contracting a blossom is an *irredundant unifactor*. Each iteration will choose the maximum weight irredundant unifactor.

For example after the contraction of Fig.11(a), the two unifactors of Fig.9 correspond to unifactors of weight $4 - W_1$ for the circuit on a and $3 - 2W_1$ for the circuit on α. The former is the maximum unifactor but the latter is the maximum irredundant unifactor. Its circuit is contracted in Fig.11(b).

We now give properties of the maximum irredundant unifactor. As the procedure forms contractions of the original graph G, it is convenient to call a vertex (or edge) of such a contraction *original* if it is the image of an original vertex (edge) of G. A vertex that is not original is the contraction of a blossom, and is called a *blossom vertex*; its corresponding (nonoriginal) loop is a *blossom loop*. Section 5.3 specifies edge weights in the contracted graph; they will be pertubed to maintain our assumption of uniqueness.

The preceding results that involve the globally maximum weight unifactor need to be extended, specifically, the second assertion of Lemma 5.3, and Lemma 5.6. For Lemma 5.3 we note that the circuit of the maximum irredundant unifactor need not be a cycle, because of blossom loops. However the original edges in the circuit form a cycle. The proof is essentially the same as Lemma 5.3. We will now prove another version of Lemma 5.4 and the extended version of Lemma 5.6 (Lemma 5.7 and Corollary 5.8).

When we contract the elementary blossom $B = (V(B), C(B), CH(B), I(B))$, the new vertex $\overline{B}$ gets degree constraint $f(\overline{B}) = |I(B)| + 1$. (See Fig.11.) The blossom loop $\overline{B}\,\overline{B}$ corresponds to an elementary blossom with odd circuit $\overline{B}\,\overline{B}$, no chords, and incident edges $I(B)$. (Clearly $f(\overline{B})$ has the required value.)

Section 5.3 gives the remaining details of how the new graph is formed when B is contracted. The current section needs just one more of these details: The weights of edges in $\delta(V(B))$ change in an unexpected way. However Lemma 5.12(ii) shows that if v is the contracted vertex for B, the maximum perturbation F_v in the new graph is the image of F_x for every $x \in V(B)$. Furthermore $F^v = F_v + vv$, so aside from the loop it is the image of each F^x, $x \in V(B)$. We use this property in the next lemma. (The proof of Lemma 5.12(ii) is simple arithmetic and does not rely on any previous lemmas.) Of course this property implies that in the new graph, $F_v + F^v$, the maximum unifactor containing v in its circuit, is the redundant unifactor with circuit vv.

Recall from the proof of Lemma 5.4 that a unifactor U whose circuit C contains v can be written as $U = U_v + U^v$ for an f_v-factor U_v and an f^v-factor U^v.

Lemma 5.7 *Let U be the maximum weight irredundant $2f$-unifactor that contains v in its circuit C, if such exists. Either $U_v = F_v$ or $U^v = F^v$.*

In Fig.11(a) the maximum irredundant unifactor containing the contracted vertex $\bar{a}$ in its circuit has the Hamiltonian circuit $\bar{a}bcdefb\bar{a}$ (the unifactor weighs $2 - 2W_1$). This illustrates the lemma with $U^{\bar{a}} = F^{\bar{a}}$ (and $U_{\bar{a}} \neq F_{\bar{a}}$).

The lemma is obvious if v is an original vertex. For a blossom vertex v, note that exactly one of the alternatives holds (since v's maximum unifactor is redundant). Furthermore $vv \notin U$ implies $U_v = F_v$, and $vv \in U - C$ implies $U^v = F^v$. The case $vv \in C$ is discussed after the proof.

Proof: Let N_v be the maximum weight f_v-factor such that $N_v + F^v$ is an irredundant $2f$-unifactor containing v in its circuit. Define N^v similarly. In general these perturbations needn't exist.

Claim *Either N_v exists and $w(U_v) \le w(N_v)$, or N^v exists and $w(U^v) \le w(N^v)$.*

Proof: $U_v + F^v$ is a $2f$-unifactor containing v in its circuit (since $U_v \oplus F^v$ contains at least one edge of $\delta(v)$). If this unifactor is redundant its circuit must be vv, and so $U_v = F_v$. Similarly if $U^v + F_v$ is redundant $U^v = F^v$. We cannot have both $U_v = F_v$ and $U^v = F_v$, since U is irredundant. So U_v or U^v gives the claim. $\diamondsuit$

By symmetry assume the claim gives N_v. Then

$$w(N_v) + w(F^v) \le w(U) = w(U_v) + w(U^v) \le w(N_v) + w(F^v).$$

We conclude equality holds throughout, so $U^v = F^v$. $\square$

Recall that when $vv \in C$ there are two choices for U_v (see the discussion after the proof of Lemma 5.4). It can be seen that one choice has $U_v = F_v$ and the other has $U^v = F^v$. (We will not use this fact.) An example is Fig.11(c): $F^{\bar{\alpha}}$ consists of loops $\bar{a}\bar{a}$ and $\bar{\alpha}\bar{\alpha}$ plus the image of ab_1. The irredundant unifactor of the figure has circuit with edges $\bar{\alpha}\bar{\alpha}, \bar{\alpha}\bar{a}, \bar{a}\bar{\alpha}$. Traversing the image of ab_1 before ab_0 gives $U_{\bar{\alpha}} = F_{\bar{\alpha}}$ while ab_0 before ab_1 gives $U^{\bar{\alpha}} = F^{\bar{\alpha}}$.

Corollary 5.8 *Suppose every maximum perturbation respects every loop blossom. Then every maximum perturbation $F\!\updownarrow_v$ respects the elementary blossom of a maximum irredundant $2f$-unifactor.*

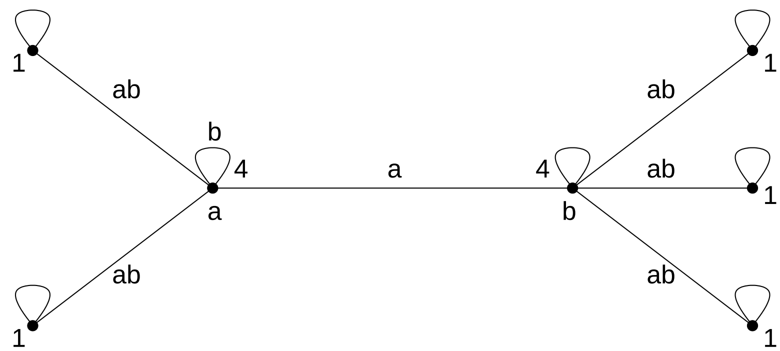


Figure 12: Example for Lemma 5.10. The edges in F_a and F_b are labelled a and b respectively. Vertex labels give degree constraints f.

Proof: As mentioned, the proof of Lemma 5.6 holds as long as every vertex of the circuit C of the unifactor satisfies (*). (The proof does not rely on the choice of unifactor U). Lemma 5.7 shows this is true. □

The circuit of the unifactor of Fig.11(c) has two original edges ab, with $ab_1 \in F_{\bar{a}} \cap F_{\bar{\alpha}}$ and $ab_0 \in F_{\bar{a}} - F_{\bar{\alpha}}$. This illustrates part (ii) of the next lemma.

Lemma 5.9 *(i) Any circuit contains either an original vertex or an edge $uv \notin F_u \oplus F_v$.*
(ii) The circuit of a maximum irredundant unifactor contains an original edge $uv \notin F_u \oplus F_v$.

Proof: Let C be a circuit in (i) or (ii). A loop vv does not belong to $F_v \oplus F_v$. So in part (i) assume C has no loops.

Take any original edge uv of C. We can assume $uv \in F_u \oplus F_v$ (if not we are done). By symmetry let $uv \in F_v - F_u$. In part (i) we can assume v is a blossom vertex (if not we are done). In part (ii) v must be a blossom vertex (if not Lemma 5.4 shows $uv \notin F_v$). The rest of the argument is identical for both parts.

Corollary 5.5(ii) shows $F_u = F^v - uv$. Let L be the set of all loops xx that are blossoms (e.g., $vv \in L$). Then

$$F_u - L - uv = F_v - L - uv. \tag{7}$$

Now let T be a trail of edges uv satisfying (7). Let T have first vertex a and first edge ab with $ab \in F_a - F_b$. We will show T cannot return to a after edge ab. Applying this to C gives a contradiction.

For an edge $uv \in T - ab$, suppose $ab \notin F_u$. Applying (7) to uv shows $ab \notin F_v$. Since $ab \notin F_b$, we conclude that $ab \notin F_u$ for every vertex $u \in T - a$. Now $ab \in F_a$ shows T cannot return to a. □

For matchings our construction terminates at the obvious point – when the graph has been contracted to a vertex. The following lemma gives the criterion for termination for f-factors. Fig.12 gives an example critical graph with no irredundant unifactor.

Lemma 5.10 *A critical graph has no irredundant $2f$-unifactor iff every vertex is a blossom vertex and every original edge uv is in $F_u \oplus F_v$ iff every circuit is the loop of a contracted blossom.*

Proof: Numbering the assertions in order as (i), (ii), (iii), we will show $(i) \implies (ii)$, $(ii) \implies (iii)$, $(iii) \implies (i)$.

$(i) \implies (ii)$: Suppose there is no noloop unifactor. A vertex that is not a loop blossom gives an irredundant unifactor (Lemma 5.4). Hence every vertex is a loop blossom. Take any original edge uv. $uv \notin F_u \cup F_v$ gives an irredundant unifactor (Corollary 5.5(i)), as does $uv \in F_u \cap F_v$ (Corollary 5.5(iii)). Thus $uv \in F_u \oplus F_v$.

$(ii) \implies (iii)$: Suppose every vertex is a blossom vertex and every original uv belongs to $F_u \oplus F_v$. Lemma 5.9(i) shows there is no circuit of original edges. A circuit that is not a contracted blossom contains a circuit of original edges.

$(iii) \implies (i)$: Obvious. □

Using Lemma 5.9(ii) in a very similar argument (like $(i) \implies (ii)$) gives the following.

Corollary 5.11 *If a maximum weight irredundant $2f$-unifactor exists then the maximum such is either $F_u + F_v + uv$ for some original edge $uv \notin F_u \cup F_v$ or $F^u + F^v - uv$ for some original edge $uv \in F_u \cap F_v$.*

5.3 The construction

This section gives the details of how the contracted graph for a blossom is formed. Iterating this process gives our complete construction for deriving the dual variables.

Given the elementary blossom $B = (C, I)$ of a maximum irredundant $2f$-unifactor U, define graph $\overline{G}$ to be G with the cycle C contracted to a vertex $\overline{C}$. So loop $\overline{C}\,\overline{C}$ exists. As usual we denote edges incident to $\overline{C}$ by their corresponding edges in G. Define degree constraints in $\overline{G}$ by

$$\bar{f}(v) = \begin{cases} f(v) & v \in V - C, \\ |I| + 1 & v = \overline{C}. \end{cases}$$

This makes $\overline{G}$ critical: For any $v \in V$ let F be an $f\updownarrow v$-factor in G that respects B (e.g. $F\updownarrow v$, Corollary 5.8). F corresponds to a subgraph $\bar{F}$ in $\overline{G}$ that is an $\bar{f}\updownarrow v$-factor for $v \in V - C$ and can be either an $\bar{f}_{\overline{C}}$-factor or an $\bar{f}^{\overline{C}}$-factor for $v \in C$. Specifically, let $\delta^+(\overline{C})$ denote the set of all edges having $\overline{C}$ as one or both ends, i.e.,

$$\delta^+(\overline{C}) = \delta(\overline{C}) + \overline{C}\,\overline{C}.$$

Then the edges of $\bar{F}$ having $\overline{C}$ as an end are

$$\bar{F} \cap \delta^+(\overline{C}) = \begin{cases} I + e & v \in V - C,\ e \text{ is some edge in } \delta(\overline{C}) - I, \text{ or} \\ I - e + \overline{C}\,\overline{C} & v \in V - C,\ e \text{ is some edge in } I; \\ I & v \in C,\ \bar{F} \text{ is an } \bar{f}_{\overline{C}}\text{-factor}, \\ I - e + \overline{C}\,\overline{C} & v \in C,\ \bar{F} \text{ is an } \bar{f}^{\overline{C}}\text{-factor}. \end{cases} \tag{8}$$

Next we define edge weights in $\overline{G}$. Observe that the first case of (8), adding an edge incident to the blossom, is similar to matching. The second case, dropping an edge, is new. Edge weights for the second case cause complications not present in matching. So to define weights $\overline{w}(e)$ in $\overline{G}$, for $v \in C$ let $c_v = w(F_v \cap \gamma(C))$, $c^v = w(F^v \cap \gamma(C))$. Let

$$J = \sum \{c^u : u \in C,\ uv \in I\} - W.$$

This sum is over a multiset, i.e., each edge $uv \in I$ contributes c^u. W is a sufficiently large number. Define

$$\overline{w}(e) = \begin{cases} w(e) & e \notin \delta(C), \\ w(e) - c^v & e \in I,\ v = C \cap e, \\ w(e) + c_v + J & e \in \delta(C) - I,\ v = C \cap e, \\ J & e = \overline{C}\,\overline{C}. \end{cases}$$

The weights in Fig.11 follow easily using $c^a = 0$ for (a), $c_b = 1$ and $c^b = 1 - 1 = 0$ for (b). In general the weights have these properties:

Lemma 5.12 *Let U be the maximum irredundant $2f$-factor of graph G, with B the elementary blossom of U. Consider the graph $\overline{G}$ with weights $\overline{w}$, formed by contracting B.*

(i) Any vertex $v \in V - C$ has a unique maximum f_v-factor and a unique maximum f^v-factor, denoted $\bar{F}_v$ and $\bar{F}^v$ respectively. They are the image of F_v and F^v respectively, and satisfy

$$\overline{w}(\bar{F}_v) = w(F_v) - W, \quad \overline{w}(\bar{F}^v) = w(F^v) - W.$$

(ii) $\overline{C}$ has a unique maximum $\bar{f}_{\overline{C}}$-factor and a unique maximum $\bar{f}^{\overline{C}}$-factor, denoted as $\bar{F}_{\overline{C}}$ and $\bar{F}^{\overline{C}}$ respectively. For any vertex $v \in C$,

$$\overline{w}(\bar{F}_{\overline{C}}) = w(F_v - \gamma(C)) - \sum\{c^u : u \in C,\ uv \in I\}, \quad \overline{w}(\bar{F}^{\overline{C}}) = w(F_v - \gamma(C)) - W.$$

Furthermore $\bar{F}^{\overline{C}} = \bar{F}_{\overline{C}} + \overline{C}\,\overline{C}$ and $\bar{F}_{\overline{C}}$ is the image of $F_v - \gamma(C) = F^v - \gamma(C)$.

(iii) Let uv be an original edge in $G - \gamma(C)$, with image $\bar{u}\bar{v}$ in $\overline{G}$ (i.e., $\bar{v}$ may be $\overline{C}$). Assume $uv \notin F_u \oplus F_v$. Let T ($\overline{T}$) be the maximum irredundant $2f$-unifactor in graph G ($\overline{G}$) that contains uv ($\bar{u}\bar{v}$) in its circuit, respectively. Then $\overline{T}$ is the image of T and

$$\overline{w}(\overline{T}) = w(T) - 2W.$$

Fig.11 illustrates the lemma: For (i) with $v = b$, in Fig.11(a) $\overline{w}(F_b) = 2 - W_1$, $\overline{w}(F^b) = 1 - W_1$ (Fig.7 gives the original weights 2 and 1). For (ii) with $C = \alpha$ and $v = b$ in (b), $\overline{w}(F_{\bar{\alpha}}) = 1 - W_1 = (1 - W_1) - 0$, $\overline{w}(F^{\bar{\alpha}}) = 1 - W_1 - W_2 = (1 - W_1) - W_2$. For (iii) with $uv = ab_1$, in (a) $w(T) = (1-1)+2-2W_1 = 2-2W_1$ and in (b) $\overline{w}(\overline{T}) = (1+(1-W_2)-W_2)-2W_1 = 2-2W_1-2W_2$.

Proof: (i): Let $\bar{F}$ be $\bar{F}_v$ or $\bar{F}^v$. The definition of W implies $\bar{F}$ contains the fewest possible number of edges in $\delta^+(C) - I$. This number is ≥ 1 ($\bar{f}(\overline{C}) = |I| + 1$). As mentioned above $F\updownarrow v$ gives a factor in $\overline{G}$ that contains exactly one of those edges. Hence $\bar{F}$ contains exactly one edge of $\delta^+(C) - I$. Since $\bar{f}(C) = I + 1$, $\bar{F}$ satisfies the first or second case of (8). Let the edge e in (8) correspond to the edge xy of G with $x \in C$.

Suppose $\bar{F}$ satisfies the first case. Its edges incident to $\overline{C}$ contribute their weight in G plus

$$c_x + J - \sum\{c^u : u \in C,\ uv \in I\} = c_x - W.$$

Since $\bar{F}$ has maximum possible weight, it is the image of the the appropriate maximum perturbation on G, i.e., that $\bar{F}_v$ is identical to $F_v - \gamma(C)$, $\overline{w}(\bar{F}_v) = w(F_v) - W$, and similarly for $\bar{F}^v$, as claimed in the lemma.

Suppose $\bar{F}$ is in the second case. The extra contribution for edges incident to $\overline{C}$ is

$$J - \sum\{c^u : u \in C,\ uv \in I - xy\} = c^x - W.$$

The rest of the argument is the same as the first case.

(ii): The proof is similar but simpler: $\bar{F}_{\overline{C}}$ must contain I and no edges of $\delta^+(\overline{C}) - I$. Each factor $F\updownarrow v$, $v \in C$, induces the same $\bar{f}_{\overline{C}}$-factor in $\overline{G}$. It contains I and it has maximum weight. So $\bar{F}_{\overline{C}}$ is the image of $F\updownarrow v - \gamma(C)$, and its weight is given in (ii).

$\bar{F}^{\overline{C}}$ must contain $I + \overline{C}\,\overline{C}$ and no other edges of $\delta^+(I)$. Again we get the weight of (ii), as well as $\bar{F}^{\overline{C}} = \bar{F}_{\overline{C}} + \overline{C}\,\overline{C}$.

(iii): If $uv \in \delta(C)$ choose v as the end in C. So we always have $u \in V - C$. Consider the two possibilities for the assumption $uv \notin F_u \oplus F_v$:

Case $uv \notin F_u \cup F_v$: This case holds iff $\bar{u}\bar{v} \notin \bar{F}_u \cup \bar{F}_v$ (parts (i) and (ii)). Corollary 5.5(i) shows T and $\overline{T}$ both exist, $T = F_u + F_v + uv$ and $\overline{T} = \bar{F}_u + \bar{F}_v + \bar{u}\bar{v}$. Thus $\overline{T}$ is the image of T as claimed.

Lastly we check the claim on weights. Suppose $uv \notin \delta(C)$. Using part (i) for u and v, plus the definition of $\overline{w}(uv)$, gives

$$\overline{w}(\overline{T}) = (w(F_u) - W) + (w(F_v) - W) + w(uv) = w(T) - 2W$$

as claimed.

Suppose $v \in C$. Using part (i) for F_u, part (ii) for $\bar{F}_{\overline{C}}$ and F_v, plus the definition of $\overline{w}(uv)$ for $uv \in \delta(C) - I$, gives

$$\begin{aligned}\overline{w}(\overline{T}) &= (w(F_u) - W) + \Big(w(F_v - \gamma(C)) - \sum\{c^u : u \in C,\ uv \in I\}\Big) + (w(uv) + c_v + J)\\ &= w(F_u) - 2W + c_v + w(F_v - \gamma(C)) + w(uv) = w(T) - 2W\end{aligned}$$

as claimed.

Case $uv \in F^u + F^v - uv$: The argument is similar to the first case. The subcase $uv \notin \delta(C)$ is as above. The equation for the second subcase is

$$\begin{aligned}\overline{w}(\overline{T}) &= (w(F^u) - W) + (w(F^v - \gamma(C)) - W) - (w(uv) - c^v)\\ &= w(F^u) + w(F^v) - 2W - w(uv) = w(T) - 2W.\end{aligned}$$

□

Our construction works by repeating the following step as long as an irredundant unifactor exists:

Create the elementary blossom B corresponding to the maximum irredundant $2f$-unifactor.
Then change the graph to $\overline{G}$ and repeat.

The following definition encapsulates the structure of the blossoms built by the procedure. (It is included mainly for possible future use, e.g., the Appendix uses it in presenting an analog of procedure *blossom_match* to construct any maximum perturbation $F\updownarrow v$ in linear time.) Consider a graph G with degree constraint function f. The definition refers to contractions of G denoted as $\overline{G}$. For notational simplicity we will not distinguish between an edge of G and its image in $\overline{G}$. Also if B is a set of one or more vertices of G, $\overline{B}$ refers to the image of B in $\overline{G}$. Finally (to motivate a set we use) note that a blossom loop $\overline{CC}$ may act like a member of $\gamma(\overline{C})$ or possibly $\delta(\overline{C})$ (e.g., $\delta^+(\overline{C})$). Because of the latter we will use a set $I(\overline{C}\,\overline{C})$ that acts like $I(C)$.

Definition 5.13 *A blossom forest $\mathcal{B}$ is a forest where each node B represents a vertex set $V(B) \subseteq V(G)$ and is labelled by three disjoint subsets of $E(G)$: $C(B), CH(B)$ and $I(B)$. The leaves of $\mathcal{B}$ are identified with the vertices of G. An interior node of $\mathcal{B}$ is called a blossom (node). For any node B, $V(B)$ is the set of leaf descendants of B.*

A leaf $v \in V$ satisfies

$$I(v) \subseteq \delta(v, G) \cup \gamma(v, G), \quad d(v, I(v)) = f(v) + 1.$$

Also $V(v) = C(v) = \{v\}$ and $CH(v) = \emptyset$.

Consider a blossom node B with children B_i, $i = 1, \ldots, k$. Form graph $\overline{G}$ as follows. If B_i is a blossom node contract it, forming vertex $\overline{B}_i$. As usual $\overline{G}$ contains the loop $\overline{B}_i\overline{B}_i$. (This holds even if $|V(B_i)| = 1$.) Set $f(\overline{B}_i) = |I(B_i)| + 1$. If B_i is a leaf corresponding to vertex v then $\overline{B}_i$ is v.

B corresponds to an elementary blossom $(\{\overline{B}_i\}, C(B), CH(B), I(B))$ in $\overline{G}$. To describe $C(B)$ we specify $C(B) \cap \overline{B}_i$, the edges of $C(B)$ incident to an arbitrary child B_i, by considering two cases:

Case $k > 1$: If B_i is a blossom then $C(B) \cap \overline{B}_i$ consists of either

(i) two edges in $I(B_i)$, or

(ii) two edges in $\delta(\overline{B}_i) - I(B_i)$, or

(iii) an edge in $I(B_i)$, loop $\overline{B}_i\overline{B}_i$, and an edge in $\delta(\overline{B}_i) - I(B_i)$.

If B_i is a leaf of $\mathcal{B}$ then (i) holds.

Case $k = 1$: B's unique child is required to be a leaf, say v. $C(B)$ is a loop vv that belongs to $I(v)$.

To specify the rest of the blossom let $I(\overline{B}_i\overline{B}_i) = \{\overline{B}_i\overline{B}_i\}$ if $k > 1$ and B_i is a blossom satisfying (i) above; in all other cases $I(\overline{B}_i\overline{B}_i) = \emptyset$. Then

$$I(B) = \bigcup_i I(B_i) \cap \delta(V(B)), \quad CH(B) = \bigcup_i (I(B_i) \cup I(\overline{B}_i\overline{B}_i)) \cap \gamma(V(B)) - C(B).$$

We give two simple properties of the definition.

Lemma 5.14 *Consider any blossom B in Definition 5.13.*

(i) $(V(B), I(B))$ is an f-odd pair.

(ii) An edge in $CH(B)$ incident to B_i belongs to $I(B_i) \cup I(\overline{B}_i\overline{B}_i)$.

Proof: (i) Any internal node B forms an elementary blossom on its children B_i, so the corresponding odd pair gives

$$f(\{B_i\}) + |I(B)| \equiv 1. \tag{9}$$

Thus $f(\{B_i\}) \equiv |I(B)| + 1 = f(B)$. So in (9), if B_i is a blossom with children C_j, we can replace the term $f(B_i)$ that contributes to $f(\{B_i\})$ by $f(\{C_j\})$. Doing this repeatedly eventually gives $f(V(B)) + |I(B)| \equiv 1$ as desired.

(ii) The equation for $CH(B)$ in Definition 5.13 allows the possibility that edges in $I(B_j)$, $j \neq i$, are chords incident to B_i. The lemma asserts this is not the case.

B is an elementary blossom so by definition, each child B_i of B satisfies

$$f(\overline{B}_i) = d(\overline{B}_i, C(B))/2 + d(\overline{B}_i, CH(B) \cup I(B)).$$

We will consider several cases. In each case every term in this equation will be known except for $d(\overline{B}_i, CH(B))$, where we only know a lower bound (from edges of $I(B_i) \cup I(\overline{B}_i\overline{B}_i)$ in $CH(B)$). We shall see that substituting all known values gives equality of the two sides. Thus the lower bound holds with equality, and no B_j has a chord incident to B_i. In what follows LHS (RHS) refer to the left- and right- hand sides of the equation, respectively.

First suppose B_i is a blossom. So $LHS = |I(B_i)| + 1$.

Suppose possibility (i) of Definition 5.13 holds. Using $d(\overline{B}_i, I(\overline{B}_i\overline{B}_i)) = 2$, the definition shows

$$RHS \geq 1 + (|I(B_i) \cap \gamma(B)| - 2) + 2 + |I(B_i) \cap \delta(B)| = 1 + |I(B_i)| = LHS.$$

If (ii) holds then

$$RHS \geq 1 + |I(B_i) \cap \gamma(B)| + |I(B_i) \cap \delta(B)| = 1 + |I(B_i)| = LHS.$$

If (iii) holds then

$$RHS \geq 2 + (|I(B_i) \cap \gamma(B)| - 1) + |I(B_i) \cap \delta(B)| = 1 + |I(B_i)| = LHS.$$

Suppose B_i is a vertex v. Then $LHS = d(v, I(v)) - 1 = 1 + (d(v, I(v) \cap \gamma(v)) - 2) + d(v, I(v) \cap \delta(v)) = RHS$. □

Now we check that the blossoms of our construction give a forest satisfying Definition 5.13. For a vertex v, the minimal blossom B containing v has $f(v) + 1$ edges incident to v in $C(B) \cup CH(B) \cup I(B)$. This gives the set $I(v)$ of the definition. It is easy to see that v satisfies the rest of the definition (i.e., property (i)).

To check an interior node B we introduce notation that will be handy later on. Recall B is created as the elementary blossom of a maximum irredundant unifactor. Let

$$\overline{G}(B), \overline{U}(B), \text{ and } \overline{V}(B)$$

denote the contraction of G wherein B is created, the unifactor, and the set of vertices of $\overline{G}(B)$ in B, respectively. So $C(B)$ is the circuit of $\overline{U}(B)$ on $\overline{V}(B)$, and $CH(B)$ ($I(B)$) is the set of chords (incident edges) of $\overline{U}(B)$ on this circuit, all at half multiplicity. These sets are defined in the construction of the elementary blossom of $\overline{U}(B)$ (Lemma 5.3), and they are the same-named sets in Definition 5.13. In that definition $\overline{V}(B)$ is denoted $\{B_i\}$ and $\overline{V}(B)$ is the image of $V(B)$ in $\overline{G}(B)$. As an example in Fig.5(b), $\overline{V}(B_1)$ has B_2 contracted to $\overline{B}_2$, and $C(B_1)$ is the triangle $\overline{B}_2, d, e$.

The discussion before and after Lemma 5.7 establishes properties (i)–(iii) of the definition for any child B_i: To repeat, letting B_i be the vertex of that discussion, $vv \notin U$ implies $U_v = F_v$ and property (ii) above holds; $vv \in U - C$ implies $U^v = F^v$ and (i) holds; if $vv \in C$ then it is easy to see (iii) holds.

Finally Lemmas 5.4 and 5.7 give the equations for $CH(B)$ and $I(B)$.

Next we examine the odd pair $(V(B), I(B))$ of Lemma 5.14(i) applied to our construction. (This pair will be used to define the z duals.)

Lemma 5.15 *The odd pair $(V(B), I(B))$ for a blossom B covers edge uv of G iff B is an ancestor of both u and v, or B is an ancestor of exactly one of u, v say v, and $uv \in F_v$.*

Proof: Clearly $u, v \in V(B)$ iff B is a common ancestor of u and v. Consider an edge uv of G whose image $\bar{u}\bar{v}$ in $\overline{G}(B)$ belongs to $I(B)$. This holds iff $\bar{u}\bar{v}$ is incident to $\bar{V}(B)$, say $\bar{v} \in \bar{V}(B) \not\ni \bar{u}$, and $\bar{u}\bar{v} \in \overline{U}(B)$. The former holds iff $v \in V(B) \not\ni u$. The latter holds iff $\bar{u}\bar{v} \in F_{\bar{v}}$. Lemma 5.12($ii$) shows this last condition is equivalent to $uv \in F_v$. □

5.4 Deriving the dual variables

We first define the dual variables. Let $\mathcal{B}$ be the final blossom forest of our construction. The term "blossom" always refers to a blossom of $\mathcal{B}$. Take any blossom B in $\mathcal{B}$. $p(B)$ denotes the parent of B in $\mathcal{B}$.

Recall that $\overline{U}(B)$ contains an original edge $uv \notin F_u \oplus F_v$ in its circuit (Lemma 5.9(ii)). So $\overline{U}(B)$ satisfies the hypothesis of Lemma 5.12(iii). Applying this lemma repeatedly shows $\overline{U}(B)$ is the image of a $2f$-unifactor in G, which we denote as $U(B)$. We illustrate by noting the weights $\overline{w}(\overline{U}(B))$ and $w(U(B))$ for Fig.11: These values are respectively 4 and 4 for (a), $3-2W_1$ and 3 for (b), $2-2W_1-2W_2$ and 2 for (c).

The reader should heed the following remark, even though it is not needed in the formal proof. As stated above $I(B)$ consists of the edges of $\overline{U}(B)$ that are incident to $\overline{V}(B)$. Equivalently $I(B)$ is the set of all edges of $U(B)$ incident to $V(B)$. But $I(B)$ need not be the set of incident edges of the elementary blossom of $U(B)$. The reason is that the circuit of $U(B)$ needn't span $V(B_i)$ for B_i a child of B. For instance in Fig.5(b), $V(B_2)=\{a,b,c\}$ and the circuit of $U(B_1)$ is $\{c,d,e\}$. (In contrast in Fig.11 for B the blossom of (c), with child blossom α, $U(B)$ spans the entirety of $\alpha=\{b,c,d,e,f\}$.)

For simplicity we refer to the odd pair $(V(B),I(B))$ as B, and we will shorten $z((V(B),I(B))$ $z(B)$. Define

$$\begin{array}{rcll} y(v) & = & -w(F_v) & v\in V, \\ z(B) & = & \begin{cases} w(U(B)) & B \text{ a root of } \mathcal{B}, \\ w(U(B))-w(U(p(B))) & B \text{ a nonroot}, \\ 0 & \text{otherwise}. \end{cases} \end{array} \tag{10}$$

Note these duals are defined on G (but not the contracted graphs $\overline{G}$). Also in this definition w denotes the given function on G with the perturbations ϵ^i omitted, i.e., we set ϵ to 0 in any context dealing with given weights. The duals of Fig.6(b) illustrate these formulas: y values come from Fig.7. z is computed using the weights $w(U(B))$ noted above from Fig.11.

Clearly any blossom B has

$$w(U(B)) = z\{C : V(B)\subseteq V(C)\}. \tag{11}$$

Observe that $z(B)\ge 0$ for any nonroot blossom B: Lemma 5.12(iii) shows $U(p(B))$ is the image of an irredundant unifactor in $\overline{G}(B)$. So it weighs less than $U(B)$ in $\overline{G}(B)$. Now applying Lemma 5.12(iii) repeatedly shows it weighs less than $U(B)$ in G, i.e., $w(U(p(B))\le w(U(B))$ as desired.

Now consider a root blossom B. $z(B)$ is certainly nonnegative if all weights in G are nonnegative. We can assume this wlog, since the number of edges in an $f\updownarrow v$-factor is fixed. Alternatively it can be seen from Lemma 5.10 that *any* $f\updownarrow v$-factor respects any root blossom of $\mathcal{B}$. (The final graph has a unique $f\updownarrow v$-factor, since the symmetric difference of two $f\updownarrow v$-factors consists of cycles.) So nonnegativity of $z(B)$ is not required in the upperbounding equation (3).

Our main theorem below refers to the optimum f-factor structure. It is defined just like the optimum matching structure, and the details are tedious. Instead of presenting them we will be content to prove that our duals are optimal for each of our maximum perturbations $F\updownarrow v$ (recall the definition of optimal duals right after equation (3).)

Theorem 5.16 *For a critical graph G, duals y,z and blossom forest $\mathcal{B}$ form an optimum f-factor structure.*

Proof: We need one more notation: For a vertex v, let B_v be the minimal blossom containing v in its circuit. The above discussion shows that in G, $U(B_v)$ is the maximum weight unifactor containing v in its circuit. So $F_v + F^v = U(B_v)$.

Let uv be an edge of G. We will verify the correct relation between $\widehat{yz}(uv)$ and $w(uv)$.

Claim 1 *The duals are tight on any edge $uv \in F_v - F_u$.*

Corollary 5.5(ii) shows $F_u = F^v - uv$. Equivalently, $F_u + F_v + uv = F^v + F_v = U(B_v)$. Taking weights shows $w(uv) = w(U(B_v)) + y(u) + y(v)$. Lemma 5.15 shows uv is covered by the ancestors of B_v and no other blossoms. So (11) implies tightness. $\diamondsuit$

Given Claim 1, from now on we assume $uv \notin F_u \oplus F_v$. Thus uv belongs to some unifactor (Corollary 5.5(i) and (iii)) and some blossom B has $u, v \in V(B)$. Let B be the nearest common ancestor of u and v in $\mathcal{B}$. uv belongs to either $C(B)$, $CH(B)$, or to no C or CH set of any blossom at all. Claims 2–4 treat these three cases. In these claims $\bar{u}$ and $\bar{v}$ denote the image of u and v in $\overline{G}(B)$, respectively.

Claim 2 *The duals are tight on any edge $uv \in C(B)$ for blossom B.*

$\overline{U}(B)$ is the maximum unifactor in $\overline{G}(B)$ containing $\bar{u}\bar{v}$ in its circuit. Applying Lemma 5.12(iii) repeatedly shows $U(B)$ is the maximum unifactor in G containing uv in its circuit. We need only check the two cases below.

Case $uv \notin F_u \cup F_v$: Corollary 5.5(i) shows $U(B) = F_u + F_v + uv$. Taking weights shows $w(U(B)) + y(u) + y(v) = w(uv)$. Lemma 5.15 shows uv is covered by the ancestors of B and no other blossoms. Applying (11) shows tightness.

Case $uv \in F_u \cap F_v$: Corollary 5.5(iii) shows $U(B) = F^u + F^v - uv$ and taking weights gives $w(U(B)) = w(F^u) + w(F^v) - w(uv)$. Adding and subtracting $w(F_u) + w(F_v)$ on the right, and rearranging, gives

$$w(uv) = w(U(B_u)) + w(U(B_v)) - w(U(B)) - w(F_u) - w(F_v).$$

Since $uv \in F_u \cap F_v$, Lemma 5.15 shows the blossoms covering uv are the ancestors of B_u or B_v (and no others). Recall B is chosen as the nearest common ancestor of u and v. Applying (11) to B_u, B_v and B shows $w(U(B_u)) + w(U(B_v)) - w(U(B)) = z\{C : \text{blossom } C \text{ covers } uv\}$. With the displayed equation this gives tightness. $\diamondsuit$

Claim 3 *The duals underrate any edge $uv \in CH(B)$ for blossom B.*

$\overline{U}(B)$ does not contain $\bar{u}\bar{v}$ in its circuit. So let $\overline{T}$ be the maximum irredundant unifactor containing $\bar{u}\bar{v}$ in its circuit in $\overline{G}(B)$. Obviously $\overline{T}$ weighs less than $\overline{U}(B)$. Lemma 5.12(iii) shows T, the maximum irredundant unifactor in G containing uv in its circuit, weighs less than $U(B)$.

The hypothesis also shows $\bar{u}\bar{v} \in F_{\bar{u}} \cap F_{\bar{v}}$, by Lemmas 5.4 and 5.7. Using Lemma 5.12(i)–(ii) repeatedly shows $uv \in F_u \cap F_v$. Corollary 5.5(iii) shows $T = F^u + F^v - uv$. Taking weights shows $w(U(B)) \geq w(T) = w(F^u) + w(F^v) - w(uv)$. As before adding and subtracting $w(F_u) + w(F_v)$ on the right and rearranging gives $w(uv) \geq w(B_u) + w(B_v) - w(U(B)) - w(F_u) - w(F_v)$. Now the argument follows Claim 2. $\diamondsuit$

Claim 4 *The duals dominate any edge $uv \notin C(B) \cup CH(B)$ for blossom B.*

(In this claim recall that B is the nearest common ancestor of u and v.) By definition $\bar{u}\bar{v} \notin \overline{U}(B)$. So let $\overline{T}$ be the maximum irredundant unifactor containing $\bar{u}\bar{v}$ in its circuit in $\overline{G}(B)$. Obviously it weighs less than $\overline{U}(B)$. Lemma 5.12(*iii*) shows T, the maximum irredundant unifactor in G containing uv in its circuit, weighs less than $U(B)$.

$\bar{u}\bar{v} \notin \overline{U}(B)$ implies $\bar{u}\bar{v} \notin F_{\bar{u}} \cup F_{\bar{v}}$, by Lemmas 5.4 and 5.7. Using Lemma 5.12(*i*)–(*ii*) repeatedly shows $uv \notin F_u \cup F_v$. Corollary 5.5(*i*) shows $T = F_u + F_v + uv$. Taking weights shows $w(U(B)) \geq w(T) = w(F_u) + w(F_v) + w(uv)$ as desired. ♢

To complete the proof take any maximum perturbation $F\updownarrow v$, $v \in V$. We will show $F\updownarrow v$ and y, z satisfy the optimality conditions of Section 5.1. $F\updownarrow v$ consists of edges in sets $C(B), CH(B)$ and $I(B)$ (Definition 5.13, Lemmas 5.4 and 5.7). These edges are underrated (Claims 1–3). The remaining edges are dominated (Claims 1,2,4).

Finally we must show $F\updownarrow v$ respects each pair $(V(B), I(B))$ of our duals. $F\updownarrow\bar{v}$ respects $(\overline{V}(B), I(B))$ in graph $\overline{G}(B)$ (Corollary 5.8, Lemma 5.12(*i*)–(*ii*)). Apply Lemma 5.1 (actually the paragraph following it) to graph $\overline{G}(B)$ to get $F\updownarrow\bar{v} \cap \delta(\overline{V}(B))$. $F\updownarrow v \cap \delta(V(B))$ is the same set, in graph G. So applying Lemma 5.1 in graph G shows $F\updownarrow v$ respects $(V(B), I(B))$.

Thus we have an optimum f-factor structure. □

Now consider a graph G that has an f-factor F. (If a vertex has $f(v) = 0$ it is irrelevant so delete it.) Let G^+ be G with an additional vertex s and additional edges sv, $v \in V$ plus loop ss, all having weight 0. Extend f by setting $f(s) = 1$. G^+ is critical: For vertex $v \in V$ with edge $vv' \in F$, $F - vv' + v's$ is an f_v-factor, $F + vs$ is an f^v-factor, F and $F + ss$ are respectively f_s and f^s -factors. So the theorem gives an optimum structured factor on G^+. Thus we have optimum duals for an f_s-factor on G^+, i.e., an f-factor. We can delete s to consider these duals defined on G. (A pair (C, I) with positive z and $z \in C$ will no longer be an odd pair. Such pairs are needed in the standard integral duals [13, Ch.32].)

Regarding uniqueness of the duals, as in previous sections the function $y = -w(F_v)$ is canonical: (4) shows that any optimal duals y, z for a critical graph satisfy $(y, z)V - y(v) = w(F_v)$. The z function is more complex and is analyzed in Section 5.5.

5.5 *b*-matchings, canonical and noncanonical duals

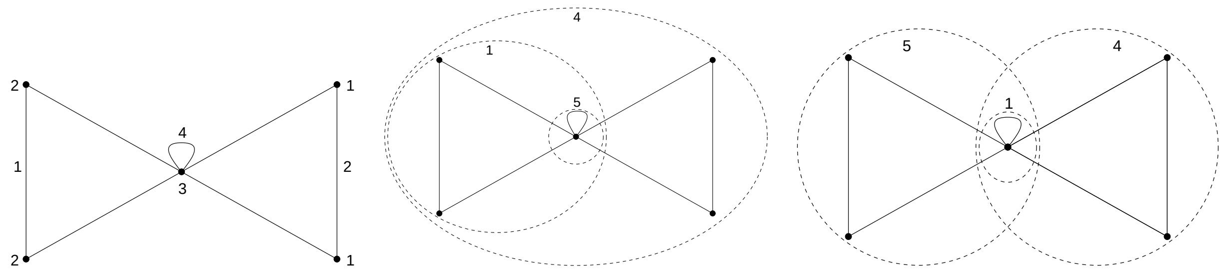
We review b-matching, a special case of f-factors, and then turn to the issue of uniqueness of the optimum duals. Section 5.4 notes that optimum duals y, z for critical graphs have y unique up to translation. This section analyzes z. It proves z is essentially unique for b-matchings. Then it presents an f-factor problem that, in contrast, has a variety of optimum z duals.

Let $G = (V, E)$ be an undirected multigraph with a function $b : V \to \mathbb{Z}_+$. A *perfect b-matching* is an assignment of a nonnegative multiplicity $\mu(e)$ to each edge e such that in the corresponding multigraph, each vertex v has degree exactly $b(v)$. When G has a weight function $w : E \to \mathbb{R}$, a *maximum b-matching* is a perfect b-matching whose total weight $\sum \mu(e)w(e)$ is as large as possible.[4]

A perfect b-matching on G corresponds to an f-factor on the multigraph constructed from G by taking $\geq \max\{b(v)\}$ copies of each edge of G. The linear program duals have two special properties [13, Ch.31]:

First there are no underrated edges in any optimum duals. This follows since any edge e in a maximum b-matching has a copy not in the b-matching. Hence e is tight.

[4]For maximum b-matchings, we could obviously assume G has no parallel edges, but loops are still allowed.



(a) Critical graph with $f \equiv 1$: Nonzero edge weights and maximum weights $w(F_v)$.

(b) Blossoms and z values.

(c) z with nonlaminar support.

Figure 13: For f-factors and b-matchings, z needn't have laminar support.

Second the z duals can be restricted to b-odd pairs of the form $(C, \emptyset)$. It is easy to see this holds for optimum duals for a critical graph: By way of contradiction consider a pair (C, I) with $v \in C$, $e \in I$, e' a copy of e not in I, and F a maximum b_v-factor. F respects (C, I) iff $F \cap \delta(C) = I$. But $F - e + e'$ is a maximum b_v-factor that does not respect (C, I).

Turning to uniqueness of z we start with the issue of laminarity. As in Section 4 an optimum z function needn't have laminar support. But Fig.5 is not a valid example for f-factors: If v is the central vertex, no f^v-perturbation respects both sets of Fig.5(c). Fig.13 remedies this. The maximum perturbations respect all pairs of (b) and (c). In both duals edge vv is tight: $\widehat{yz}(vv) = 2(-3) + 10 = 4 = w(vv)$. The example is valid for b-matchings as well as f-factors.

We will show that assuming laminarity makes z unique for b-matching. We start with two properties of f-factors. The properties hold *a fortiori* for b-matchings and they will be used in the b-matching analysis. The properties will also give intuition for the f-factor example.

Let ET be the set of edges of G that are always tight,

$$ET = \{e : \widehat{yz}(e) = w(e) \text{ for every pair of optimal duals } y, z\}.$$

(w is the unperturbed weight function and y, z are any duals that are optimal for it.)

Lemma 5.17 *Let C be the circuit of an elementary blossom formed in some graph $\overline{G}$. C contains at most one edge not in ET. If such an edge uv exists then it is the unique original edge satisfying $uv \in C - (F_u \oplus F_v)$, and C consists entirely of blossom vertices.*

Proof: Any edge $e \in F \oplus F'$ for two maximum perturbations F, F' must be both underrated and dominated by any optimal dual function. Hence $e \in ET$.

If C contains an original vertex v of $\overline{G}$ then every edge of C is in $F_v \oplus F^v$ and hence it is tight. So the lemma holds in this case. (In particular the lemma holds when C is a loop blossom vv.)

Suppose C consists entirely of blossom vertices. Lemma 5.9(ii) shows C contains an original edge $uv \notin F_u \oplus F_v$. If $uv \notin F_u \cup F_v$ Corollary 5.5(i) implies every every edge of $C(B) - uv$ is in $F_u \oplus F_v$, and hence it is tight. Corollary 5.5(iii) gives the same conclusion if $uv \in F_u \cap F_v$.

We conclude $C - ET$ contains at most one edge. Furthermore if uv is such an edge then any original edge $xy \in C - uv$ is in $F_x \oplus F_y$, since otherwise the argument of the previous paragraph makes uv tight. $\square$

Clearly the lemma shows that when $uv \in C - ET$, every nonloop edge $xy \in C - uv$ belongs to $I(x) \oplus I(y)$ (any nonloop edge is original).

Let EF be the set of edges of G belonging to some maximum perturbation,

$$EF = \bigcup\{F_v \cup F^v : v \in V\}.$$

(As usual F_v and F^v denote the unique maximum weight subgraphs that we have defined.) Let $\mathcal{Z}$ be the support of the z function of an arbitary pair of optimum duals.

Lemma 5.18 *Any $\mathcal{Z}$-set is connected by the edges of EF.*

Proof: Suppose Z is the union of disjoint sets Z_0, Z_1 with no edge of EF joining them. Let I_0 be the edges of I incident to Z_0 and similarly for I_1. Take $v \in Z_1$. The maximum perturbation F_v respects Z, so $F_v \cap \delta(Z) = I$. Thus for $i = 0, 1$, any edge of F_v incident to a vertex of Z_i is in $\gamma(Z_i) \cup I_i$. This makes $f(Z_0) + |I_0|$ even and $f(Z_1) + |I_1|$ odd. Choosing $v \in Z_0$ gives the opposite. $\Box$

We now prove the canonical duals are essentially unique for b-matching. For a dual function y, z and a set of vertices S, let

$$\bar{z}(S) = z\{X : S \subseteq X\}.$$

We use this notation for two types of sets S: edges (those being sets of two vertices) and sets in the support of z (especially when the support is laminar).

Let $\mathcal{Z}$ be the support of z. Note that $\widehat{yz}(e) = y(e) + \bar{z}(e)$ since $I = \emptyset$ on $\mathcal{Z}$. So $e \in ET$ implies $\bar{z}(e) = w(e) - y(e)$ for any optimal duals, i.e., $\bar{z}(e)$ does not depend on the choice of duals. Note also that if $\mathcal{Z}$ is laminar and edge $e \subseteq Z \in \mathcal{Z}$, then Z is the minimal $\mathcal{Z}$-set containing e iff $\bar{z}(e) = \bar{z}(Z)$.

As before, let G be a critical graph for b-matching, let y, z be an arbitrary set of optimal duals and let y^*, z^* denote our duals.

Theorem 5.19 *If the support of z is laminar and $z(G) = z^*(G)$ then $z = z^*$.*

Proof: Let $\mathcal{Z}$ ($\mathcal{Z}^*$) be the support of z (z^*). We will show $\mathcal{Z} = \mathcal{Z}^*$ and $\bar{z} = \bar{z}^*$. Obviously laminarity then proves $z = z^*$ as desired. The bulk of the argument treats the nonsingleton sets of $\mathcal{Z}$ and $\mathcal{Z}^*$.

Lemma 5.18 shows any nonsingleton $\mathcal{Z}$-set contains an edge $e \in EF$. For b-matching the root blossom of $\mathcal{B}$ contains every vertex (Lemma 5.10). So $e \in C(A) \cup CH(A)$ for some blossom A.

Claim 1 *Let Z be the smallest $\mathcal{Z}$-set containing edge $e \in EF$. If $e \in C(A) \cup CH(A)$ for blossom A then $A \subseteq Z$.*

Suppose $A \not\subseteq Z$. Let A' be a minimal blossom contained in A that crosses Z. (Possibly $A' = A$.) $C(A')$ contains (at least) two edges crossing Z, so some $f \in C(A') \cap ET$ crosses Z (Lemma 5.17). Laminarity of $\mathcal{Z}$ implies

$$\bar{z}(f) < \bar{z}(e)$$

and $A' \subseteq A$ implies

$$\bar{z}^*(f) = \bar{z}^*(A') \geq \bar{z}^*(A) = \bar{z}^*(e).$$

But $f \in ET$ and $e \in EF \subseteq ET$ give $\bar{z}(f) = \bar{z}^*(f)$, $\bar{z}(e) = \bar{z}^*(e)$. This makes the two displayed inequalities contradictory. $\diamondsuit$

Claim 2 *In Claim 1 choose e and A so A is maximal for Z. Then $Z = A \in \mathcal{Z}^*$ and $\bar{z}(Z) = \bar{z}^*(A)$.*

By Claim 1 $A \subseteq Z$. Let $f \in EF$ leave A. We can assume $f \in C(D) \cup CH(D)$ for some blossom D (since as already noted the root blossom of $\mathcal{B}$ contains every vertex, Lemma 5.10). Since $A \subset D$ maximality implies f leaves Z. Lemma 5.18 shows $A = Z$.

Any edge $f \in EF \cap \delta(A)$ has

$$\bar{z}^*(f) = \bar{z}(f) < \bar{z}(Z) = \bar{z}(e) = \bar{z}^*(A).$$

Thus A is a maximal blossom with $\bar{z}^*$ value $\bar{z}^*(A)$, i.e., $A \in \mathcal{Z}^*$. This also shows $\bar{z}(Z) = \bar{z}^*(A)$ (even if f does not exist). $\diamondsuit$

We now show the desired conclusion holds for nonsingleton sets, i.e., $\mathcal{Z}$ and $\mathcal{Z}^*$ contain the same nonsingletons, and $\bar{z}$ and $\bar{z}^*$ agree on these sets. Claim 2 shows a nonsingleton of $\mathcal{Z}$ is in $\mathcal{Z}^*$. Conversely any nonsingleton $B \in \mathcal{Z}^*$ is the smallest blossom containing any edge $e \in C(B) \cap EF$. If Z is the smallest $\mathcal{Z}$-set containing e, Claim 1 shows $B \subseteq Z$. Z does not contain any edge $f \in EF \cap \delta(B)$, since

$$\bar{z}(f) < \bar{z}^*(B) = \bar{z}^*(e) = \bar{z}(e) = \bar{z}(Z).$$

Thus $Z = B$. So the nonsingletons of $\mathcal{Z}^*$ belong to $\mathcal{Z}$.

We complete the proof by analyzing the singleton sets. Subtracting (4) from (4′) shows $\sum\{z(C,I) : (C,I) \in \mathcal{I},\ v \in C\}$ does not depend on the choice of optimum duals. In other words any $v \in V$ satisfies $\bar{z}(vv) = \bar{z}^*(vv)$. Since $\bar{z}$ and $\bar{z}^*$ agree on nonsingletons this implies $z(vv) = z^*(vv)$, so $\mathcal{Z}$ contains vv iff $\mathcal{Z}^*$ does. $\Box$

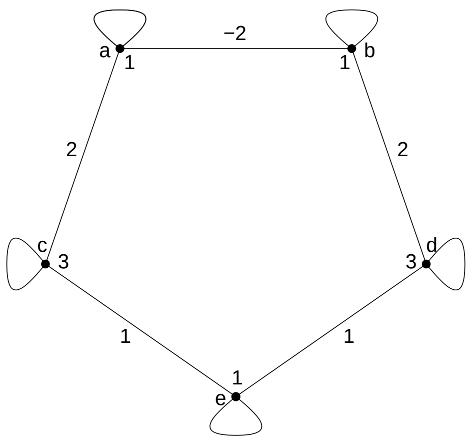
We turn to the f-factor problem. Fig.14 gives a graph G where laminar optimum duals have different z functions. We now explain the example.

Fig.14(a) shows G, f, and the edge weights. Note that each vertex has a weight 0 loop. To check that G is critical it suffices to check each vertex v is in the cycle of a $2f$-unifactor. Fig.14(b) shows the loop vv is the cycle of such a unifactor. For instance the unifactor for a consists of loop aa and the 4 edges labelled by a, each with multiplicity 2. The unifactor for b is symmetric. Similarly for the other vertices.

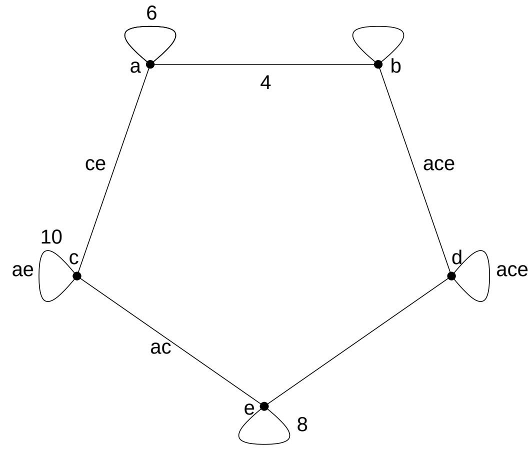
In addition to these five "loop" unifactors there are unifactors based on the 5-cycle $acedba$. An even number of loops can be added, but the weight of the unifactor remains 4. The weights of all unifactors are shown in Fig.14(b). These weights give the maximum unifactor containing each vertex and each edge, justifying the dual values shown in Fig.14(c).

Fig.14(c) shows the sets $I(cc)$ and $I(dd)$ each contain two edges and all other blossoms have empty I sets. So if C is the 5-cycle, every nonloop edge $xy \in C - ab$ belongs to $I(x) \oplus I(y)$. We show below that $ab \notin ET$. In other words our example is fashioned after the nontight alternative in Lemma 5.18, and the remark following it. In fact the reader will see that alternate duals like those in Fig.14(d) and (e) are easily constructed for any similar blossom.

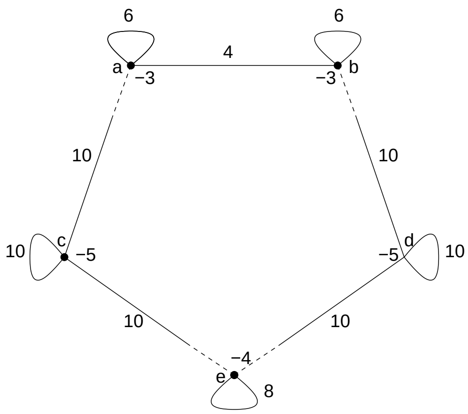
Fig.14(d) gives the blossoms and their z values, computed from (10). Fig.14(e)–(f) give alternative optimum z functions. As mentioned they are based on the fact that every edge besides ab is covered by exactly one loop blossom. This allows us to decrease z on loop blossoms, increasing z on compensating blossoms. Fig.14(f) does this for loop blossoms cc, dd, ee. Fig.14(e) does this for all the loop blossoms. Here edge ab changes from tight to strictly dominated, $\widehat{yz}(ab) = 0 > -2$. This is permissible since ab is not in any maximum weight perturbation. (In proof, a perturbation containing ab has ≤ 1 weight 2 edge. Thus it weighs ≤ 2. The y values in Fig.14(c) show this is never maximum.)



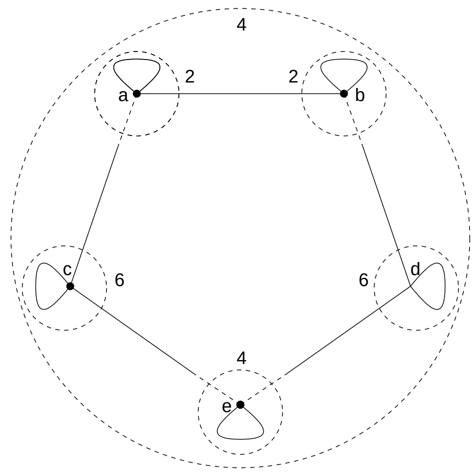
(a) Graph, f and nonzero edge weights.



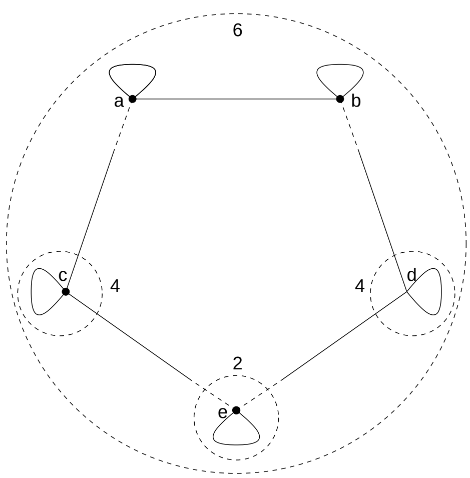
(b) Unifactors for aa, cc, ee and the 5-cycle: Weights and multiplicity 2 edges.



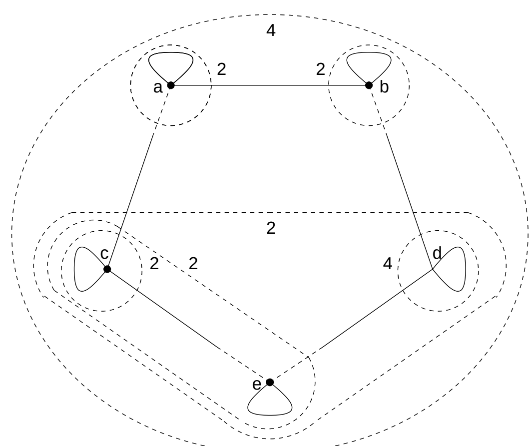
(c) $y(v)$ and $\bar{z}^*(e)$.



(d) Blossoms and z^*-values.



(e) z with smaller support.



(f) Nonblossoms in the support.

Figure 14: f-factors do not have unique z duals.

6 Conclusions

We present several precise questions for further research.

1. Can the combinatorial interpretation be used algorithmically? For example determinant-based algorithms for matching are well-known [6, 9, 12]. The adjoint matrix can be used to get the weights $w(M_v)$ that give y dual variables (and the ζ values for z in general matching). Reducing processor counts in parallel algorithms is one likely application.

2. The problems treated in this paper have many close relatives [13]. Do their dual variables have similar combinatorial interpretations? Matroid intersection and matroid parity are good candidates.

3. Blossom-based algorithms for maximum weight matching [5] find the maximum weight unifactor as their maximum weight elementary blossom. Can this unifactor be found faster?

A Finding an f_v-factor

This appendix presents an algorithm that returns the f_v- or f^v-factor specified by a blossom tree $\mathcal{B}$ (given $\mathcal{B}$ and the desired perturbation, f_v or f^v for an arbitrary vertex v). By definition this perturbation respects every blossom of $\mathcal{B}$. Hence it will have maximum weight for the blossom tree constructed in Section 5. The algorithm operates in linear time. It is based on Definition 5.13.

Initialize the desired subgraph F to contain $\bigcup\{I(v)\} - \bigcup\{C(B)\}$, where v and B range over all leaf nodes and blossom nodes of $\mathcal{B}$ respectively. Then execute the procedure $factor(R, v, a)$, where R is the root node of $\mathcal{B}$ and a is 1 (-1) if we seek an f^v- (f_v-) factor, respectively. It will enlarge F to the desired perturbation by adding the edges in circuits $C(B)$ for blossoms B in $\mathcal{B}$.

In general the recursive procedure $factor(B, x, a)$ is called with B a blossom node of $\mathcal{B}$, $x \in V(B)$, and $a = \pm 1$. For $a = 1$ $(a = -1)$ it adds the circuit edges of an f^x- (f_x-) perturbation to F. Let B have children B_i in $\mathcal{B}$, with $x \in V(B_1)$. The procedure starts at B_1 and traverses the circuit $C(B)$, adding alternate edges of $C(B)$ to F. The rule for choosing the edge incident to B_1 that begins the traversal is given below. Before this we indicate how each blossom B_i, $i \neq 1$, is processed. This depends on which of the possibilities (i) – (iii) of Definition 5.13 hold for B_i.

If (i) holds, one of the two edges of $I(B_i)$ is not added to F; let it be rs with r in B_i. If B_i is a leaf, i.e., vertex r, this completes the processing of vertex r. Otherwise call $factor(B_i, r, 1)$.

If (ii) holds, one of the two edges not in $I(B_i)$ is added to F; let it be rs with r in B_i. Call $factor(B_i, r, -1)$.

If (iii) holds, the two nonloop edges of $\delta(\overline{B}_i)$ are either both added or both not added to F. If they are not added then proceed as in possibility (i), else proceed as in possibility (ii).

Now we describe how the traversal begins at B_1. If B_1 is the leaf x then possibility (i) holds. The parameter a indicates whether or not the two edges of $I(x)$ (or the loop at xx, if $k = 1$) should be added to in F. Thus the starting edge incident to x is chosen (arbitrarily) and the traversal begins by adding it to F if $a = 1$, else not adding it. The traversal ends by processing the other edge incident to x similarly.

Next suppose B_1 is a blossom. In all cases the traversal adds all edges of $C(B) \cap I(B_1)$ to F, and it ends by calling $factor(B_1, x, a)$. Further details depend on which possibility (i) – (iii) holds for B_1. If (i) holds, the traversal starts by adding one of the edges of $I(B_1)$ to F; it ends by adding the other to F. If (ii) holds, the traversal starts by following one of the edges not in $I(B_1)$, but it is not added to F; the traversal ends by processing the other edge incident to B_1 similarly. If (iii) holds, the traversal starts by adding the edge of $I(B_1)$ to F; it ends by traversing the edge not in $I(B_1)$ then traversing the loop $\overline{B}_1\overline{B}_1$, adding neither.

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