Problem 1  (25 pts)

Recall that the trace of a matrix $A$, denoted $\text{tr}(A)$, is the sum of the entries along the diagonal.

1. Prove that if $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

2. Prove that if $A$ is a random walk matrix of an $n$-vertex graph $G$ and $k \geq 1$, then $\text{tr}(A^k)$ is equal to $n$ times the probability that if we select a vertex $v \in V(G)$ uniformly at random and take a $k$ step random walk from vertex $v$, then we end up back at vertex $v$.

3. Prove that for every $d$-regular graph $G, k \in \mathbb{N}$ and vertex $v \in V(G)$ of $G$, the probability that a path of length $k$ starting from $v$ ends up back at $v$ is at least as large as the corresponding probability in $T_d$, where $T_d$ is the complete $(d-1)$-ary tree of depth $k$ rooted at $v$ (that is, every internal vertex has degree $d$, one parent and $d-1$ children).
Problem 2  (25 pts)

1. Prove that if $M$ is the transition matrix of a regular undirected graph $G$ and $\lambda_1 \geq \cdots \geq \lambda_n$ are its eigenvalues with multiplicities, then the number of eigenvalues equal to 1 is the same as the number of connected components of $G$. Hint: If $\lambda$ is an eigenvalue of $M$, then the set of vectors $x$ such that $Mx = \lambda x$ forms a linear space. For the solution of this problem you can assume the following result: the multiplicity of $\lambda$ is the same as the dimension of linear space $\{x : Mx = \lambda x\}$.

2. Let $G$ be an undirected regular graph, $M$ be its transition matrix, $\lambda_1 \geq \cdots \geq \cdots \geq \lambda_n$ be the eigenvalues of $M$. Prove that $\lambda_n = -1$ if and only if $G$ is bipartite.

3. Let $G$ be an undirected regular graph, $M$ be its transition matrix, $\lambda_1 \geq \cdots \geq \cdots \geq \lambda_n$ be the eigenvalues of $M$. Prove that

$$\max_i |\lambda_i| = \max_{x \in \mathbb{R}^n} \frac{||Mx||}{||x||}$$

Problem 3  (25 pts)

Let BIPARTITE denote the language of all (undirected) graphs which are bipartite. Show that BIPARTITE is in NL.