CSCI 7000-005 Computational Complexity
Problem Set 3

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To be worked during class for the week of October 8-12, 2018. Due October 18.

Collaboration Policy: The homework can be worked on in groups of up to 3 students each (2 would be optimal, but 1 and 3 are both accepted).

One submission per team is sufficient. Please write the solution for each of the problems on a separate sheet of paper. Write your team's names and id on each submission and please **staple** all the sheets together.

Submissions should be written in LaTeX, unless your handwriting is indistinguishable from LaTeX.

Homework is due before the end of class, October 18. Only one late homework per person will be allowed. If you submit more than one homework late, you will get no grade for the excess late homeworks.

Problem 1 (25 pts)

Recall that the trace of a matrix A, denoted $\mathrm{tr}(A)$, is the sum of the entries along the diagonal.

1. Prove that if $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \dots, \lambda_n$, then $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

2. Prove that if A is a random walk matrix of an n-vertex graph G and $k \geq 1$, then $\mathrm{tr}(A^k)$ is equal to n times the probability that if we select a vertex $v \in V(G)$ uniformly at random and take a k step random walk from vertex v, then we end up back at vertex v.

3. Prove that for every d-regular graph $G, k \in \mathbb{N}$ and vertex $v \in V(G)$ of G, the probability that a path of length k starting from v ends up back at v is at least as large as the corresponding probability in T_d, where T_d is the complete $(d-1)$-ary tree of depth k rooted at v (that is, every internal vertex has degree d, one parent and $d-1$ children).

Problem 2 (25 pts)

1. Prove that if M is the transition matrix of a regular undirected graph G and $\lambda_1 \geq \cdots \geq \cdots \geq \lambda_n$ are its eigenvalues with multiplicities, then the number of eigenvalues equal to 1 is the same as the number of connected components of G. Hint: If λ is an eigenvalue of M, then the set of vectors x such that $Mx = \lambda x$ forms a linear space. For the solution of this problem you can assume the following result: the multiplicity of λ is the same as the dimension of linear space $\{x : Mx = \lambda x\}$.

2. Let G be an undirected regular graph, M be its transition matrix,$\lambda_1 \geq \cdots \geq \cdots \geq \lambda_n$ be the eigenvalues of M. Prove that$\lambda_n = -1$ if and only if G is bipartite.

3. Let G be an undirected regular graph, M be its transition matrix,$\lambda_1 \geq \cdots \geq \cdots \geq \lambda_n$ be the eigenvalues of M. Prove that

$$\max_i |\lambda_i| = \max_{x \in \mathbf{R}^n} \frac{||Mx||}{||x||}$$

Problem 3 (25 pts)

Let BIPARTITE denote the language of all (undirected) graphs which are bipartite. Show that BIPARTITE is in NL.